E1.1 Circuit Analysis

Problem Sheet 6 - Solutions

1. (a) Negative, (b) Positive, (c) Negative, (d) Positive, (e) Positive.

2. (a) The time constant is $\frac{1}{100} = 10\,\text{ms}$. (b) We need to solve $5.5 = 5 + 2e^{-100t} \Rightarrow e^{-100t} = 0.25 \Rightarrow -100t = \ln 0.25 = -1.386 \Rightarrow t = 13.86\,\text{ms}$. Alternatively, we can use the standard formula, derived in lectures, $t = \tau \ln\left(\frac{7-5}{5.5-5}\right) = 10 \times \ln 4 = 13.86\,\text{ms}$. (iii) The general formula, derived in lectures, is $T_{A\to B} = \tau \ln\left(\frac{A-5}{B-5}\right)$.

3. The current is $I = \frac{-200j}{4+5j-2j} = \frac{-200j}{4+3j} = -24 - 32j = 40\angle - 127°$. So $\left|\widetilde{I}\right|^2 = \frac{40^2}{2} = 800$. The complex power absorbed by each of the passive components is $\left|\widetilde{I}\right|^2 Z$; this gives $\left|\widetilde{I}\right|^2 R = 3.2\,\text{kW}$, $\left|\widetilde{I}\right|^2 Z_L = 4000j = 4\,\text{kVAR}$ and $\left|\widetilde{I}\right|^2 Z_C = -1600j = -1.6\,\text{kVAR}$. The current through the source (following the passive sign convention) is $-I = 24 + 32j$ and the complex power absorbed by it is $\widetilde{V}\left(-\widetilde{I}\right)^* = -141j\,(17 - 22.6j) = (-3.2 - 2.4j)$ kVA. As expected, the total complex power sums to zero.

4. (a) The DC gain of the circuit is 1, so the steady state output is $x_{SS}(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$. Because x is the voltage across a capacitor, it must be continuous, so $x(0+) = x(0-) = 0$. So the complete expression is $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 5 - 5e^{\frac{-t}{\tau}}$ where $\tau = RC$. (b) $x(t)$ is plotted in Fig. 4.

   (c) Using the standard formula, $t = \tau \ln\left(\frac{0-5}{4.5-5}\right) = \tau \ln(10) = 2.3RC$.

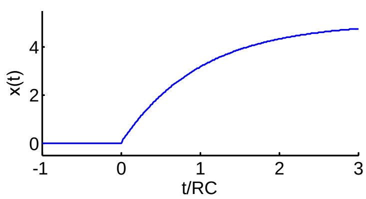

Fig. 4

5. (i) From the circuit, the time constant is $RC = 2\,\text{ms}$. The DC gain may be obtained by treating C as an open circuit and is 0; the HF gain may be obtained by treating C as a short-circuit and is therefore 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{j\omega RC}{1+j\omega RC}$ which happily gives the same values: $\frac{Y}{X}(0) = 0$, $\frac{Y}{X}(\infty) = 1$, $\tau = \frac{1}{\text{denominator corner frequency}} = RC$. The importance of $\frac{Y}{X}(\infty)$ is that it gives the gain for a step input discontinuity, i.e. $\frac{Y}{X}(\infty) = \frac{\text{output discontinuity}}{\text{input discontinuity}}$.

   (ii) From the circuit: $\tau = RC = 2\,\text{ms}$, DC gain (C open-circuit) = 1, HF gain (C short-circuit) = 0. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{1}{1+j\omega RC}$ which gives the same values.

   (iii) From the circuit: $\tau = \frac{L}{R} = 0.1\,\text{ms}$, DC gain ($L$ short-circuit) = 0, HF gain (L open-circuit) = 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{j\omega L}{R+j\omega L}$ which gives the same values. Note that if the denominator is $(p + j\omega q)$, the time constant is $\frac{q}{p}$ and the corner frequency is $\frac{p}{q}$.

   (iv) To obtain the time constant, we need to determine the Thévenin resistance seen by the inductor. To do this, we set the input voltage, X, to zero (thereby shorting node X to ground) and find the resistance between the inductor terminals (with the inductor removed). The two resistors are in series, so we get $R_{Th} = 1 + 4 = 5\text{k}$. From this, $\tau = \frac{L}{R} = 200\,\text{ns}$, DC gain ($L$ short-circuit) = 0.2 (potential divider), HF gain (L open-circuit) = 1. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{R+j\omega L}{5R+j\omega L}$ where $R = 1\,\text{k}$. This gives the same values and is, perhaps, a marginally easier way to determine them.

(v) To determine the Thévenin resistance seen by the capacitor, we set $X = 0$ and measure the resistance at the capacitor terminals. Since X is connected to ground, the two $1\,\text{k}$ resistors are in series and so we have $8\,\text{k}$ in parallel with $2\,\text{k}$ which gives $1.6\,\text{k}$. From this, $\tau = RC = 0.16\,\text{ms}$, DC gain ($C$ open-circuit) $= 0.1$ (potential divider), HF gain (C short-circuit) $= 0.5$. The transfer function (using potential divider formula) is $\frac{Y}{X}(j\omega) = \frac{R}{2R+\frac{8R}{1+8j\omega RC}} = \frac{1+8j\omega RC}{10+16j\omega RC}$ where $R = 1\,\text{k}$ and we used the formula for $Z_{8R||C} = \frac{8R}{1+j\omega 8RC}$.

(vi) To determine the Thévenin resistance at the capacitor terminals is not trivial because of the dependent voltage source that is the opamp. If we set $X = 0$ and replace the capacitor with a voltage source V as in Fig. 5(i), we can use nodal analysis to determine I and then calculate $R_{Th} = \frac{V}{I}$. Since the op-amp is a unit-gain buffer, $Y = V$. KCL at node W gives: $\frac{W}{10} + \frac{W-V}{10} + \frac{W-V}{10} = 0 \Rightarrow W = \frac{2}{3}V \Rightarrow I = \frac{V-W}{10} == \frac{\frac{1}{3}V}{10} = \frac{V}{30} \Rightarrow R_{Th} = 30\,\text{k}$. So, finally, we get $\tau = R_{Th}C = 3\,\text{ms}$. For the DC gain, we make C an open-circuit as in Fig. 5(ii). Negative feedback means $V_+ = Y$ and, since there is no current through the resistor connected to V_+, we must also have $W = Y$. KCL at node W then gives $W = Y = X$, so the DC gain is 1. For the HF gain, the capacitor acts a a short circuit so $V_+ = 0$ which in turn means that $Y = 0$ so the gain is 0.

Rather easier is the transfer function approach. We know $V_+ = Y$ and V_+ is determined from W by an RC potential divider giving: $\frac{Y}{W} = \frac{V_+}{W} = \frac{1}{1+j\omega RC}$. KCL at W gives $\frac{W-X}{R} + \frac{W-Y}{R} + \frac{W-Y}{R} = 0 \Rightarrow 3W - 2Y = X$. We now substitute for W using the previous equation $\frac{Y}{W} = \frac{1}{1+j\omega RC}$ to get $3Y(1+j\omega RC) - 2Y = X \Rightarrow \frac{Y}{X} = \frac{1}{1+3j\omega RC}$. From this we can easily get: $\tau = 3RC$, $\frac{Y}{X}(0) = 1$ and $\frac{Y}{X}(\infty) = 0$.

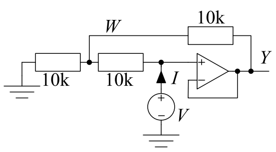

Fig. 5(i)

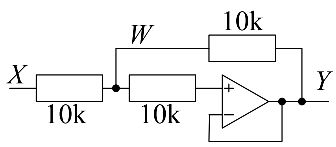

Fig. 5(ii)

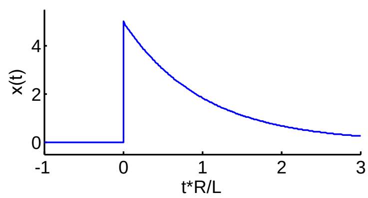

Fig. 7

6. (i) $v(t)$ equals 6 for $\frac{1}{3}$ of the time and -2 for $\frac{2}{3}$ of the time. So its average value is $\overline{v} = \frac{1}{3}\times 6 + \frac{2}{3}\times(-2) = \frac{2}{3}$. Similarly, the average value of v^2 is $\overline{v^2} = \frac{1}{3}\times 36 + \frac{2}{3}\times 4 = 14\frac{2}{3}$. So $V_{rms} = \sqrt{14.67} = 3.83$. This is higher than the average value $\overline{v}$.

   (ii) During the first period ($0 \le t \le 2$), the formula for v can be derived as $v = 2t$. To find the average value, we integrate over one period, and divide by the length of the period. So $\overline{v} = \frac{1}{2}\int_{t=0}^{2} 2t dt = \frac{1}{2}\left[t^2\right]_0^2 = 2$. This is also pretty obvious from looking at the waveform. In the same way, $\overline{v^2} = \frac{1}{2}\int_{t=0}^{2}(2t)^2 dt = \frac{1}{2}\left[\frac{4}{3}t^3\right]_0^2 = 5\frac{1}{3}$ giving $V_{rms} = \sqrt{5.33} = 2.31$.

   (iii) This is the same as the previous waveform but shifted up by $+2$. You can perform integrations similar to the previous part or, easier, just modify the previous answers. $\overline{v}$ (which previously equalled 2) will be increased by 2 to become $\overline{v} = 4$. Adding a constant onto a random variable does not affect its variance, so $\overline{v^2} - (\overline{v})^2$ will be unchanged at $5\frac{1}{3} - 2^2 = 1\frac{1}{3}$. It follows that $\overline{v^2} = (\overline{v})^2 + 1\frac{1}{3} = 17\frac{1}{3}$. Taking the square root gives $V_{rms} = \sqrt{17.33} = 4.16$.

7. Method 1 (inductor current continuity): For $t < 0$, $x = 0$ and the current through the inductor is $i = \frac{v-x}{R} = 0$. It follows that at time $t = 0+$, the current through the resistor (which equals the current through the inductor) will still be zero and $x(0+) = v(0+) = 5$. From this value is will decay to a steady state value $x_{SS} = 0$ since the inductor is a short circuit for DC. Thus, $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 0 + 5e^{\frac{-t}{\tau}}$ where $\tau = \frac{L}{R}$. This is plotted in Fig. 7.

   Method 2 (transfer function): The transfer function of the circuit is (from potential divider equation) $\frac{X}{V} = \frac{j\omega L}{R+j\omega L}$. From this we get the DC gain, $G_{DC} = 0$, the HF gain, $G_{HF} = 1$, and the time constant is $\frac{L}{R}$. The DC gain allows us to calculate the steady state $x_{SS}(t) = G_{DC}v(t) \equiv 0$. The output discontinuity at $t = 0$ is given by $\Delta x = G_{HF}\Delta v = 1 \times 5 = 5$. So $x(0+) = x_{SS}(0-) + \Delta x = 0 + 5 = 5$. Finally we put everything together to get: $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 0 + (5 - 0)\, e^{\frac{-t}{\tau}}$.

8. Method 1 (capacitor voltage continuity): The time constant of the circuit is obtained by setting $v = 0$ and finding the Thévenin resistance across the capacitor terminals. Since v is connected to ground, the two resistors are in parallel and $R_{Th} = \frac{1}{2}R$ giving $\tau = \frac{1}{2}RC$. The DC gain of the circuit is 0.5, so $x_{SS}(t) = 0.5v(t) = \begin{cases} 1 & t < 0 \\ 3 & t \geq 0 \end{cases}$. For $t < 0$, the capacitor voltage is $v - x = 2 - 1 = 1$. This must remain continuous and so $v(0+) - x(0+) = 1 \Rightarrow x(0+) = v(0+) - 1 = 5$. Putting everything together, we get $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 3 + (5 - 3)\, e^{\frac{-t}{\tau}} = 3 + 2e^{\frac{-t}{\tau}}$. This is plotted in Fig. 8.

   Method 2 (transfer function): The transfer function of the circuit is (from potential divider equation) $\frac{X}{V} = \frac{R}{R + \frac{R}{1+j\omega RC}} = \frac{1+j\omega RC}{2+j\omega RC}$. From this we get the DC gain, $G_{DC} = 0.5$, the HF gain, $G_{HF} = 1$, and the time constant is $0.5RC$. The DC gain allows us to calculate the steady state as above. The output discontinuity at $t = 0$ is given by $\Delta x = G_{HF}\Delta v = 1 \times 4 = 4$. So $x(0+) = x_{SS}(0-) + \Delta x = 1 + 4 = 5$. Finally we put everything together to get: $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 3 + (5 - 3)\, e^{\frac{-t}{\tau}} = 3 + 2e^{\frac{-t}{\tau}}$.

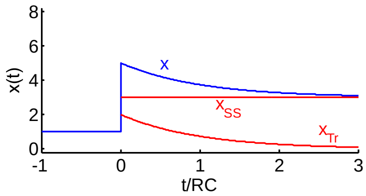

Fig. 8

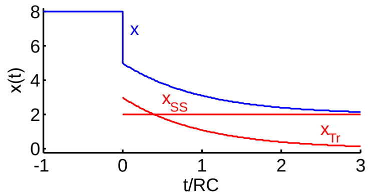

Fig. 9

9. For opamp circuits, it is easiest to use the transfer function to determine the relevant circuit parameters. This is a non-inverting amplifier with a gain of $\frac{X}{V} = 1 + \frac{\frac{R}{1+j\omega RC}}{R} = \frac{2+j\omega RC}{1+j\omega RC}$. Thus we have a DC gain, $G_{DC} = 2$, a high frequency gain $G_{HF} = 1$ and a time constant $\tau = RC$. At $t = 0$, the input discontinuity is $\Delta V = v(0+) - v(0-) = -3$ and so the ouput discontinuity is $\Delta X = x(0+) - x(0-) = G_{HF}\Delta V = 1 \times -3 = -3$. The steady state output is given by $x_{SS}(t) = G_{DC}v(t) = \begin{cases} 8 & t < 0 \\ 2 & t \geq 0 \end{cases}$. So this gives $x(0+) = -3 + x(0-) = -3 + 8 = 5$.

   Putting everything together, we get $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 2 + (5 - 2)\, e^{\frac{-t}{\tau}} = 2 + 3e^{\frac{-t}{RC}}$.

10. (a) Average power is $\frac{\widetilde{V}^2}{R_L}$, so $R_L = \frac{\widetilde{V}^2}{10\,\text{k}} = 5.76\,\Omega$. (b) Current through R_L is $\widetilde{I}_L = \frac{\widetilde{V}}{R} = 41.7\,\text{A}$. The current through R_S is $\frac{\widetilde{I}_L}{n}$ so the power dissipation is $\frac{\widetilde{I}_L^2 R_S}{n^2}$. This gives (i) 868 W and (ii) 34.8 W.

11. (a) Impedances are transformed by the square of the turns ratio (because the voltage decreases by n and the current increases by n). So when $n = 4$, the impedance at the output of the secondary is $\frac{2400}{16} = 150\,\Omega$.

    (b) At 50 Hz, the capacitor impedance is $Z_C = \frac{1}{2\pi 50 \times 100\,\text{n}} = -j31.8\,\text{M}\Omega$. With the transformed source impedance from part (a), we get the equivalent circuit shown in Fig. 11. Using superposition, $V_A = V_N \times \frac{Rn^{-2}}{Rn^{-2} + Z_C} + V_S n^{-1} \times \frac{Z_C}{Rn^{-2} + Z_C}$ so the ratio of the signal and noise voltage magnitudes is $\frac{V_S n^{-1}|Z_C|}{V_N R n^{-2}} = \frac{V_S n |Z_C|}{V_N R}$. This gives (i) 35.2 dB for $n = 1$ and (ii) 47.3 dB for $n = 4$.

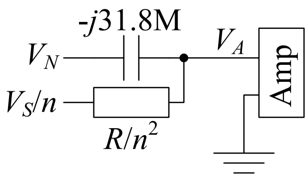

Fig. 11

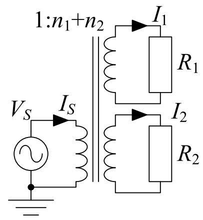

Fig. 12

12. (a) $\widetilde{V}_1 = n_1\widetilde{V}_S = 2$ so the average power dissipated in R_1 is $\frac{\widetilde{V}_1^2}{R_1} = 400\,\text{mW}$. Similarly, $\widetilde{V}_2 = n_2\widetilde{V}_S = 3$, so the average power dissipated in R_2 is $\frac{\widetilde{V}_2^2}{R_2} = 450\,\text{mW}$.

(b) From the ideal transformer equations, $1\times I_S + n_1\times(-I_1) + n_2\times(-I_2) = 0$ (the minus signs arise because in Fig. 12 I_1 and I_2 are defined as coming out of the transformer). Rearranging this and using Ohm's law gives $I_S = n_1I_1 + n_2I_2 = \frac{n_1V_1}{R_1} + \frac{n_2V_2}{R_2} = \frac{n_1^2V_S}{R_1} + \frac{n_2^2V_S}{R_2}$. So $R_{eff} = \frac{V_S}{I_S} = \frac{1}{\frac{1}{n_1^{-2}R_1} + \frac{1}{n_2^{-2}R_2}}$. This is the same as the parallel combination of $n_1^{-2}R_1$ and $n_2^{-2}R_2$, i.e. the parallel combination of the individual winding resistances transferred from the secondaries to the primary.

13. (a) The inductor impedance is $j\omega L = j\times 100\pi\times 0.008 = 2.51\,\Omega$. So the current is $I = \frac{V_S}{R+j\omega L} = 41.5-65.1j$. So $S = VI^* = P+jQ = 9.54+14.98j\,\text{kVA}$. So the apparent power is $|S| = 17.8\,\text{kVA}$and the average and reactive powers are P and Q given earlier. The power factor is $\cos\phi = \frac{P}{|S|} = 0.54$.

(b) Adding the capacitor will not consume any average power and so will not affect P at all. We need to reduce Q to $P\tan(\arccos 0.9) = 4.62\,\text{kVAR}$ since $\tan\phi = \frac{Q}{P}$ and we want $\cos\phi = 0.9$. It follows that $Q_C = 4.62 - 14.98 = -10.36\,\text{kVAR} = \frac{-|V_S|^2}{|Z_C|} = -230^2\omega C$. This gives $C = 623\,\mu\text{F}$. Now $S = P + jQ = 9.54 + 4.62j\,\text{kVA}$. So the apparent power is $|S| = 10.6\,\text{kVA}$ and the average and reactive powers are P and Q given earlier. The power factor is $\cos\phi = \frac{P}{|S|} = 0.9$.

14. Notice first that since there are no input discontinuities, there will be no output discontinuities either. The circuit transfer function is $\frac{X}{V} = \frac{j\omega RC}{1+j\omega RC}$. The gain at $\omega = 2000\pi$ is $G = 0.503 + 0.5j = 0.709\angle 44.8°$. The phasor corresponding to $v(t) = 5\sin\omega t$ is $V = -5j$ and so the steady state output will be $X = GV = 2.5 - 2.51j = 3.54\angle - 45.2°$ which corresponds to a waveform $x_{SS}(t) = 2.5\cos\omega t + 2.51\sin\omega t$. The time constant is $RC = 0.16\,\text{ms}$.

(a) At time $t = 0+$ we have $x_{SS}(0+) = 2.5\,\text{V}$. Since there is no output discontinuity, $x(0+) = x(0-) = 0$. Putting this together gives $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\,e^{\frac{-t}{\tau}} = 2.5\cos\omega t + 2.51\sin\omega t + (0 - 2.5)\,e^{\frac{-t}{RC}}$. This is plotted in Fig. 14(i).

(b) Substituting $t = 1$ into the previous expression gives $x(1-) = x(1+) = 2.495$ (very close to the steady state value since it has had $6\frac{1}{4}$ time constants to converge). For $t > 1$, the steady state is $x_{SS}(t) \equiv 0$. Therefore we get $x(t) = 2.495e^{\frac{-(t-1)}{\tau}}$. This is plotted in Fig. 14(ii).

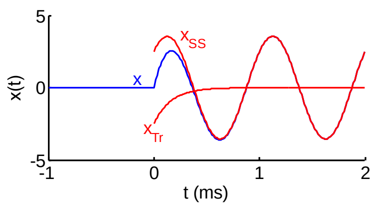

Fig. 14(i)

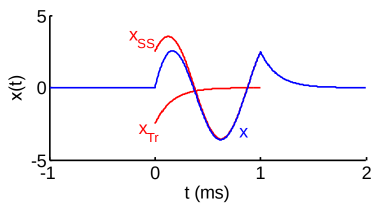

Fig. 14(ii)

15. For the steady state when $t < 0$, we can treat the inductor as a short circuit and so $i = \frac{10}{R} = 100\,\text{mA}$. When the switch is closed, there is a constant $10\,\text{V}$ across the inductor and so $\frac{di}{dt} = \frac{V}{L} = 100\,\text{V/s}$. Therefore the current through the inductor will increase linearly at this rate for $2\,\text{ms}$ (from an initial value of $100\,\text{mA}$) and will reach a value of $300\,\text{mA}$. Since there is no resistor in series with the inductor, the current increases linearly rather than exponentially; you can, if you wish, regard this as a limiting case of a negative exponential that has an infinite time constant.

When the switch is opened at $t = 2$ ms, the current will decay from its peak value of 300 mA back down to its steady state value of 100 mA with a time constant of $\frac{L}{R} = 1$ ms. Thus for $t > 2$ ms, we have $i(t) = 100 + (300 - 100)\, e^{\frac{-(t-2)}{1}}$ (in units of mA and ms). All this is plotted in Fig. 15(i).

When the switch is open, $v(t) = Ri(t)$. However, when the switch is closed, $v(t) \equiv 0$. We therefore get the voltage waveform plotted in Fig. 15(v).

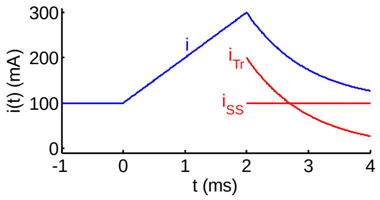

Fig. 15(i)

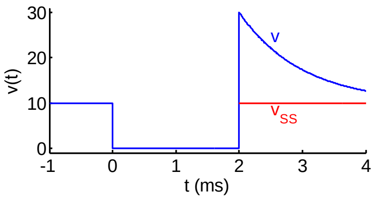

Fig. 15(v)

16. In this question, we have two different circuits according to whether the diode is off or on. These are shown in Fig. 16(off),(on). When the diode is off, we have a DC steady state $x_{SS}(t) = 0$ and a time constant $\tau_{Off} = RC = 1.6$ ms. On the other hand, when the diode is on, we can get the DC steady state by doing KCL for the shaded supernode: $\frac{x+0.7-v}{2} + \frac{x}{8} = 0 \Rightarrow x_{SS} = \frac{4}{5}(v - 0.7)$. We obtain the time constant by setting all voltage sources to zero and finding the Thévenin resistance and the capacitor terminals: this is $R_{Th} = 2\,\text{k}||8\,\text{k} = 1.6\,\text{k}$; this gives a time constant $\tau_{On} = 0.32$ ms.

For $t < 0$, $v = x = 0$ and so, since $v - x < 0.7$, the diode will be off. When v changes to 3, the diode will turn on and will charge the capacitor up to a steady state voltage of $\frac{4}{5}(3 - 0.7) = 1.84$. When v now changes to 2 V, the diode will turn off and x will fall towards the "off" steady state of 0 V. However, it will never reach this value, because when x reaches $v - 0.7 = 1.3$ V the diode will turn on again resulting in a new steady state of $\frac{4}{5}(2 - 0.7) = 1.04$ V. So this means we actually have four distinct time segments: $t < 0$, $0 \le t < 1$, $1 \le t < T_x$, $t \ge T_x$ wher T_x is the, as yet unknown, time at which the diode turns on for the last time.

Segment 1 (Diode Off, $t < 0$, $x = v = 0$).

Segment 2 (Diode On, $0 \le t < 1$, $v = 3$, $x_{SS} = \frac{4}{5}(3 - 0.7) = 1.84$, $\tau_{On} = 0.32$ ms): $x(t) = x_{SS}(t) + (x(0+) - x_{SS}(0+))\, e^{\frac{-t}{\tau}} = 1.84 + (0 - 1.84)\, e^{\frac{-t}{\tau}} = 1.84 - 1.84e^{\frac{-t}{\tau_{On}}}$. At $t = 1$ this gives $x(1) = 1.76$.

Segment 3 (Diode Off, $1 \le t < T_x$, $v = 2$, $x_{SS} = 0$, $\tau_{Off} = 1.6$ ms): Capacitor voltage continuity means that $x(1+) = 1.76$. So $x(t) = x_{SS}(t) + (x(1+) - x_{SS}(1+))\, e^{\frac{-t}{\tau}} = 0 + (1.76 - 0)\, e^{\frac{-t}{\tau}} = 1.76e^{\frac{-(t-1)}{\tau_{Off}}}$. We need to know when the voltage x reaches 1.3 V ($t = T_x$) because that is when the diode will turn on again. Solving $1.76e^{\frac{-(T_x-1)}{\tau_{Off}}} = 1.3 \Rightarrow T_x = 1.48$ ms.

Segment 4 (Diode On, $t \ge 1.48$ ms, $v = 2$, $x_{SS} = \frac{4}{5}(2 - 0.7) = 1.04$, $\tau_{On} = 0.32$ ms): $x(t) = x_{SS}(t) + (x(T_x+) - x_{SS}(T_x+))\, e^{\frac{-t}{\tau}} = 1.04 + (1.3 - 1.04)\, e^{\frac{-t}{\tau}} = 1.04 + 0.26e^{\frac{-(t-T_x)}{\tau_{On}}}$.

All four segments are plotted in Fig. 16(iii).

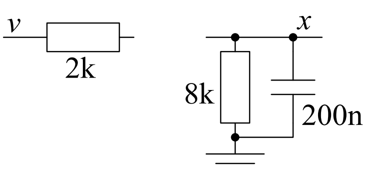

Fig. 16(off)

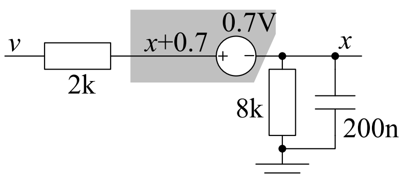

Fig. 16(on)

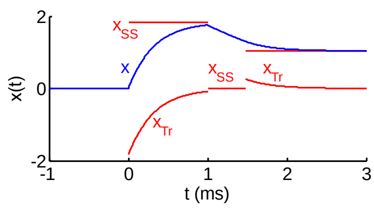

Fig. 16(iii)