E1.1 Circuit Analysis

Problem Sheet 7 - Solutions

1. The propagation velocity (from page 17-4 of the notes) is $u = \sqrt{\frac{1}{L_0 C_0}} = 2 \times 10^8$ m/s which is $\frac{2}{3}$ of the speed of light. The characteristic impedance is $Z_0 = \sqrt{\frac{L_0}{C_0}} = 100\,\Omega$. These are typical characteristics for the twisted pair cable used in computer networks.

2. The reflection coefficients is given by $\rho = \frac{R - Z_0}{R + Z_0}$ where R is the Thévenin impedance at the relevant end of the line. This gives (a) $\rho_S = -0.818$, $\rho_L = 0$ so a pulse at V_S will travel down the line as a forward wave and stop when it reaches the load (no reflections), (b) $\rho_S = -0.818$, $\rho_L = 0.818$ so a positive pulse at V_S will travel down the line as a forward wave, be reflected at $x = L$ and travel back towards the source as a backward wave. When it reaches $x = 0$ it will be reflected and inverted and will result in a negative pulse traveling as a forward wave. This whole process will be repeated for ever and you will get an infinite sequence of pulses each one smaller than than the previous one, (c) $\rho_S = 0$, $\rho_L = 0.818$ so a positive pulse at V_S will travel down the line as a forward wave, be reflected at $x = L$ and then travel back towards $x = 0$ where it will stop.

3. Since the voltage source is DC, $f()$ and $g()$ will be constants and the waves will be independent of x and t. From Ohm's law, we know that the voltage and current in the line are 10 V and 0.2 A so we must have $f + g = 10$ and $f - g = 0.2 \times 100 = 20$. Solving these two equations gives $f = 15$ and $g = -5$. As we would expect, the ratio $\frac{g}{f} = -0.333$ is equal to the reflection coefficient at the load. The power carried by the two waves is $\frac{f^2}{Z_0} = 2.25$ W and $\frac{g^2}{Z_0} = 0.25$ W and the difference between these does indeed equal the power absorbed by the load: $\frac{10^2}{50} = 2$ W.

4. At $x = 0$, $v_0(t) = f(t) + g(t)$ and $i_0(t) = Z_0^{-1}\left(f(t) - g(t)\right)$ which gives the waveforms shown in Fig. 4(a). The peak voltage and current are 9 V and 90 mA respectively. At $x = 300$ cm, $f\left(t - \frac{x}{u}\right)$ is delayed by 20 ns while $g\left(t + \frac{x}{u}\right)$ is advanced by the same amount. The new voltage and current waveforms are as shown in Fig. 4(b). The forward and backward waves now overlap and the peak voltage and current are 12 V and 60 mA respectively.

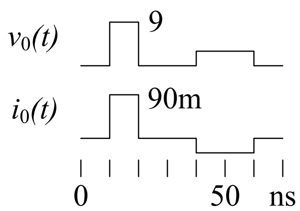

Fig. 4(a)

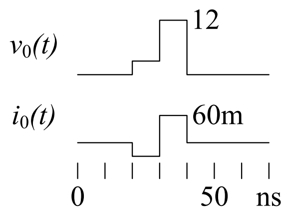

Fig. 4(b)

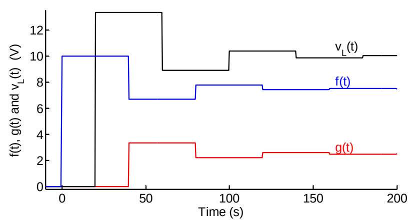

Fig. 5

5. (a) $\rho_L = \frac{100-50}{100+50} = 0.33$, $\rho_0 = \frac{0-50}{0+50} = -1$.
(b) In the steady state we must have $V = 10$ V and $I = 0.1$ A which means that $f + g = 10$ and $f - g = 0.1 Z_0 = 5$. We can solve these two equations to give $f = 7.5$ and $g = 2.5$.
(c) From the notes (page 17-11) the forward wave is the sum of an infinite number of copies of the input signal; each extra copy is delayed by an additional round-trip propagation delay ($\frac{600}{15} = 40$ ns) and multiplied by an additional factor $\rho_L \rho_0 = -0.33$. This is shown in Fig. 5; the input signal jumps to 10 V when the switch is closed and stays there forever. Onto this is added a copy of the input signal delayed by 40 ns and multiplied by -0.33 which means that $f(t)$ jumps to 6.67 at $t = 40$ ns. With pulse waveforms, it is sometimes easier to keep track of the *changes* in the signals rather than their absolute values. In this case, the input signal only changes once: by +10 V at $t = 0$. It follows that $f(t)$ will have a change of +10 at $t = 0$ followed by a change of $10\rho_L\rho_0 = -3.33$ at $t = 40$ followed by a change of $10\left(\rho_L\rho_0\right)^2 = 1.11$ at $t = 40$ followed by a change of $10\left(\rho_L\rho_0\right)^3 = -0.37$ at $t = 60$ and so on.
As we can see, $f(t) \to 7.5$ as predicted in part (a).
$g(t) = \rho_L f(t - \frac{2L}{u})$ is just the same as $f(t)$ but delayed by the round-trip propagation delay and multiplied by $\rho_l = 0.33$.

The voltage waveform at $x = 0$ is equal to $f(t) + g(t)$ and has a constant value of 10 V for $t \geq 0$ as is obvious from the circuit.
The voltage waveform at $x = L$ is $v_L(t) = f_L(t) + g_L(t) = f(t-20) + g(t+20)$, that is, the sum of $f(t)$ delayed by 20 ns and $g(t)$ advanced by 20 ns. This reaches a peak value of $10\,(1 + \rho_L) = 13.3$ V.

6. Note that the propagation velocity in SI units is 20×10^7 m/s. At 20 MHz the wavenumber is $k = \frac{\omega}{u} = \frac{2\pi \times 2 \times 10^7}{20 \times 10^7} = 0.628$ rad/m or, equivalently, the wavelength is $\lambda = \frac{u}{f} = \frac{2\pi}{k} = 10$ m. From the notes (page 18-8) the impedance of a short-circuit stub of length L is $Z_T = jZ_0 \tan kL$ which leads to $L = \frac{n\lambda}{2} + \frac{1}{k}\tan^{-1}\frac{Z_T}{jZ_0}$ where the $\frac{n\lambda}{2}$ term arises because tan() repeats every π. We want to choose L so that (a) $Z_T = \frac{1}{j\omega C} = -159j$ or (b) $Z_T = j\omega L = 126j$ and in each case choose n to make L as small as possible while still remaining positive. This gives (a) $L = 3.39$ m and (b) $L = 1.43$ m.
To derive the formula for Z_T, we first note that the reflection coefficient at the short circuit is $\rho_L = -1$ and that F travels down the line, is reflected at the short circuit and then travels back along the line to become G, that is $G = F \times e^{-kL} \times -1 \times e^{-kL}$ where e^{-kL} is the phase shift due to the time it takes for the wave to travel a distance L. Thus $G = -Fe^{-2kL}$ and the impedance of the line is $Z_T = \frac{V}{I} = \frac{F+G}{Z_0^{-1}F - G} = Z_0\frac{1-e^{-2kL}}{1+e^{-2kL}} = Z_0\frac{e^{kL}-e^{-kL}}{e^{kL}+e^{-kL}} = Z_0\frac{2j\sin kL}{2\cos kL} = jZ_0\tan kL$.

7. (a) The wavelength is $\lambda = \frac{u}{f} = \frac{20 \times 10^7}{50 \times 10^6} = 4$ m which means that we get a phase shift of $-\frac{\pi}{2}$ for every metre we move along the line. Equivalently $k = \frac{\omega}{u} = \frac{2\pi}{\lambda} = 1.571$ rad/m. Thus at $x = 0, 1, 2, 3, 4, 5$, $F_x = F_0 e^{-jkx} = 6j, 6, -6j, -6, 6j, 6$. Notice that the phase repeats every 4 m since this is one wavelength.
(b) The reflection coefficient at $x = L$ is $\rho_L = \frac{R_L - Z_0}{R_L + Z_0} = -0.333$.
(c) It follows that $G_L = \rho_L F_L = -0.333 \times 6 = -2$. As with F_x in part (a), G_x will get a phase shift of $-\frac{\pi}{2}$ for every metre we move backwards along the line towards $x = 0$, i.e. $G_x = G_L e^{-jk(L-x)}$. If you substitute for G_L you can write this as $G_x = \rho_L F_L e^{-jk(L-x)} = \rho_L F_0 e^{-jkL} e^{-jk(L-x)} = \rho_L F_0 e^{-jk(2L-x)}$. Thus at $x = 0, 1, 2, 3, 4, 5$, $G_x = 2j, -2, -2j, 2, 2j, -2$.
(d) $V_x = F_x + G_x$, so at $x = 0, 1, 2, 3, 4, 5$, $V_x = 8j, 4, -8j, -4, 8j, 4$. We can see that when F_x and G_x are in phase (at $x = 0, 2, 4$) their magnitudes add to give $|V_x| = 6 + 2 = 8$ whereas when they are out of phase (at $x = 1, 3, 5$) the subtract to give $|V_x| = 6 - 2 = 4$.
(e) From part (d), the VSWR is $\frac{8}{4} = 2$. Because $|G_x| = |\rho_L|\,|F_x|$, the VSWR is $\frac{1+|\rho_L|}{1-|\rho_L|}$ which, if you express ρ_L in terms of R_L equals $\max\left(\frac{R_L}{Z_0}, \frac{Z_0}{R_L}\right) = \max(0.5, 2) = 2$.
(f) The line impedance at $x = 0$ is $\frac{V_0}{I_0} = Z_0\frac{F_0+G_0}{F_0-G_0} = Z_0\frac{8j}{4j} = Z_0 \times 2 = 200$.

8. (a) The forward wave is unaffected so $F_x = F_0 e^{-jkx} = 6j, 6, -6j, -6, 6j, 6$ as before.
(b) The line termination is matched and the reflection coefficient is now 0.
(c) It follows that $G_x = 0\,\forall x$.
(d) $V_x = F_x + G_x = F_x + 0 = F_x$, so $V_x = 6j, 6, -6j, -6, 6j, 6$.
(e) $|V_x|$ is always equal to 6 and so the VSWR equals 1, its minimum possible value. Measuring the VSWR gives a way of telling when a line is matched without having to measure either Z_0 or R_L.
(f) The line impedance at $x = 0$ is $\frac{V_0}{I_0} = Z_0\frac{F_0+G_0}{F_0-G_0} = Z_0\frac{F_0+0}{F_0-0} = Z_0 = 100$.

9. (a) We have $\lambda = 7.5$ m and $k = \frac{2\pi}{\lambda} = 0.838$. The reflection coefficient is $\rho_L = \frac{R_L - Z_0}{R_L + Z_0} = 0.2$. Hence $\rho(0) = \frac{G_0}{F_0} = \rho_L e^{-2jkL} = -0.0209 - 0.1989j$.
(b) From the notes (page 18-7), $Z_T = Z_0\frac{1+\rho(0)}{1-\rho(0)} = 100 \times (0.8874 - 0.3677j) = 88.74 - 36.77j = 96.06\angle - 22.5°$.
(c) Hence $V_0 = V_S\frac{Z_T}{R_S + Z_T} = 10 \times (0.911 - 0.033j) = 9.11 - 0.33j$.
(d) Since $V_0 = F_0 + G_0 = F_0\,(1 + \rho(0))$, we can write $F_0 = \frac{V_0}{1+\rho(0)} = 9.00 + 1.49j$. From this we get $F_L = F_0 e^{-jkL} = 7.13 - 5.69j$. We can now calculate $G_L = \rho_L F_L = 1.43 - 1.14j$ and then work out $V_L = F_L + G_L = 8.56 - 6.83j$. Notice that once you know F and G at one point on the line, you can easily calculte their values at any other point whereas V varies in a rather more complicated manner; because of this, we convert from V to F, then move F to a different place and convert back to V again. As shown in the notes (on page 18-7 but beware that there was a error in the original version distributed) you can do this algebraically and get a formula that does the whole thing in one go: $V_L = \frac{1+\rho_L}{1+\rho(0)} e^{-jkL} V_0$.

(e) The complex power supplied by the source is

$$\frac{\left|\tilde{V}_S\right|^2}{(R_S+Z_T)^*}=\frac{\frac{1}{2}\left|V_S\right|^2}{(R_S+Z_T)^*}=\frac{50}{98.7+36.8j}=0.445-0.166j \text{ VA}$$

. You might expect this to be purely real because the circuit appears to contain only resistors; however, there are implicit capacitors and inductors in the transmission line.