Paper Number(s): **E1.1**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2009

EEE/ISE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Wednesday, 10 June 10:00 am

There are FOUR questions on this paper.

Q1 is compulsory.
Answer Q1 and any two of questions 2-4.
Q1 carries 40% of the marks. Questions 2 to 4 carry equal marks (30% each).

Time allowed: 2:00 hours

Examiners responsible:

First Marker(s): D.M. Brookes D.M. Brookes

Second Marker(s): P.D. Mitcheson P.D. Mitcheson

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$.
2. Component and source values in a circuit are normally given in ohms, farads, henrys, volts or amps with the unit symbol omitted. Where an imaginary value is given, it represents the complex impedance.

1. (a) Using nodal analysis calculate the voltage at node X in *Figure 1.1.* [5]

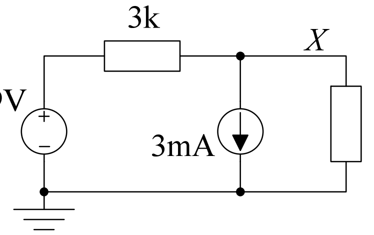


Figure 1.1

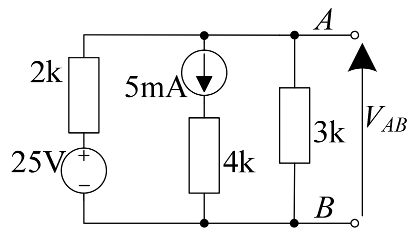


Figure 1.2

(b) Use the principle of superposition to find the voltage V_{AB} in *Figure 1.2.* [5]

(c) Draw the Thévenin equivalent circuit of the network in *Figure 1.3* and find the values of its components. [5]

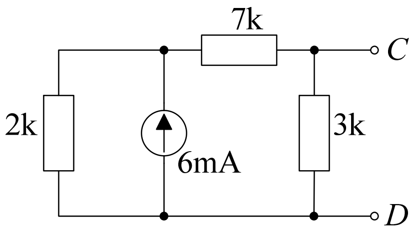


Figure 1.3

(d) Assuming that the diode in *Figure 1.4* has a forward voltage drop of $0.7\,\text{V}$, determine the voltage at node Y when the voltage at X is (i) $+5\,\text{V}$ and (ii) $-5\,\text{V}$. [5]

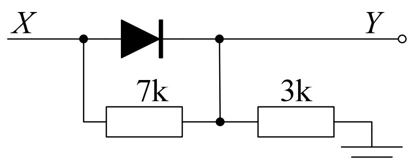


Figure 1.4

(e) The phasor representing the voltage at X in *Figure 1.5* has the value $2j$. Calculate the phasor representing the voltage at Y and express the waveform at Y in the form $y(t) = A\cos(\omega t) + B\sin(\omega t)$. Components are labelled with their complex impedances. [5]

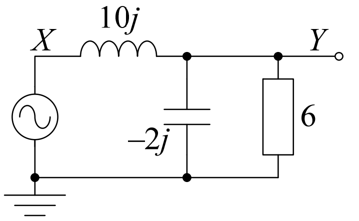


Figure 1.5

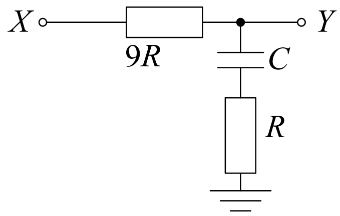


Figure 1.6

(f) Calculate the frequency response, $\frac{Y}{X}(j\omega)$, of the network shown in *Figure 1.6*. Determine the gain at high frequencies and sketch a dimensioned graph of the magnitude response in dB versus frequency using a log frequency scale. If $RC = 0.001$, determine the corner frequencies in rad/s. [5]

(g) In the circuit of *Figure 1.7*, the r.m.s. phasor $\tilde{X}$ has the value 100. Determine the value of the phasor current $\tilde{I}$ and the complex power $\tilde{V} \times \tilde{I}^*$ absorbed by each of the three components. Component values represent complex impedances. [5]

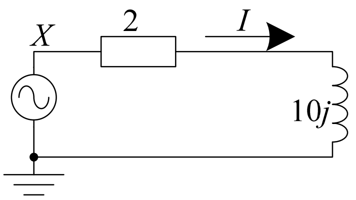


Figure 1.7

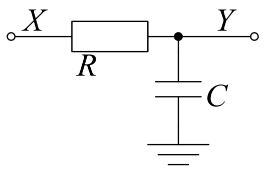


Figure 1.8

[5]

(h) In *Figure 1.8*, the input voltage, $x(t) = \begin{cases} 5 & t < 0 \\ 10 & t \geq 0 \end{cases}$.

(i) Calculate the steady state output voltages at Y for $t < 0$ and for $t \geq 0$.

(ii) Determine an expression for $y(t)$ for $t \geq 0$.

2. (a) Assuming that the op-amp in the circuit of *Figure 2.1* is ideal, show that the gain is given by $\frac{Y}{X} = \frac{Z_2(Z_3 + Z_4)}{Z_3(Z_1 + Z_2)}$. [6]

(b) Calculate the transfer function, $\frac{Y}{X}(j\omega)$, for the circuit of *Figure 2.2* expressing your answer as a ratio of two factorized polynomials in $j\omega$. [6]

(c) Draw a dimensioned sketch of the magnitude response $\left|\frac{Y}{X}(j\omega)\right|$ for the circuit of *Figure 2.2*. Calculate the value of all corner frequencies and the gain in dB of all horizontal portions of the response. [14]

(d) If $R = 10\,\text{k}\Omega$ in the circuit of *Figure 2.2*, calculate the value of C so that the lowest corner frequency is at $300\,\text{Hz}$. [4]

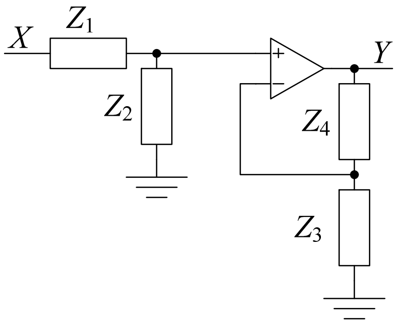


Figure 2.1

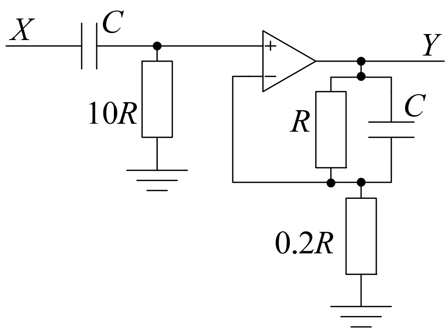


Figure 2.2

3. The circuit of *Figure 3.1* represents a voltage source driving a load of $Z_L = (6+8j)\Omega$ via a line having impedance $(1.2+1.6j)\Omega$. The supply voltage is $240\,\text{V}_{\text{rms}}$ at $50\,\text{Hz}$ and is represented by the r.m.s. phasor $\widetilde{X} = 240$.

(a) Calculate the current in the circuit, $\widetilde{I}$, as an r.m.s. phasor and hence calculate the average power absorbed by the line and by the load. [12]

(b) Determine whether the current lags or leads the supply voltage and the phase angle between them. [4]

(c) Calculate the admittance, $\dfrac{1}{Z_L}$, of the load and hence determine the value of capacitor (in Farads) that should be placed between terminals A and B so that the combination of load and capacitor is purely resistive. [6]

(d) With the capacitor in place, calculate the current now supplied by the voltage source and the average power dissipated in the line. [8]

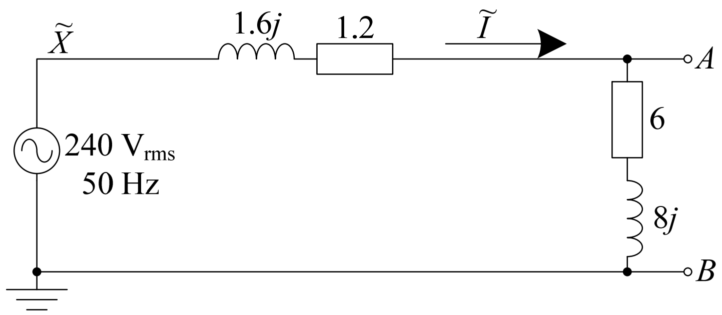


Figure 3.1

4. In the circuit of *Figure 4.1*, the input voltage is X and the output is Y.

(a) Show that the time constant of the circuit is $0.8RC$. [4]

(b) If the input voltage is given by $x(t)=\begin{cases}0 & t<0\\ 10 & t\geq 0\end{cases}$

(i) Calculate the steady-state values at Y for $t<0$ and for $t\geq 0$. [5]

(ii) Calculate the transient amplitude and hence find an expression for $y(t)$ for $t\geq 0$. [5]

(c) Now suppose that the input voltage is given by $x(t)=\begin{cases}0 & t<0\\ \sin(\omega t) & t\geq 0\end{cases}$ where $\omega=\frac{2}{RC}$.

(i) Show that the transfer function of the circuit is $\frac{Y}{X}(j\omega)=\frac{1+4j\omega RC}{5+4j\omega RC}$. [5]

(ii) Determine the steady-state AC component of $y(t)$ for $t\geq 0$ in the form $A\cos(\omega t)+B\sin(\omega t)$. [6]

(iii) Calculate the transient amplitude and hence find an expression for $y(t)$ for $t\geq 0$. [5]

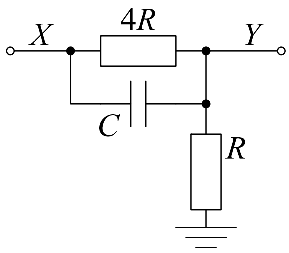


Figure 4.1

2009 E1.1: Analysis of Circuits - Solutions

Key to letters on mark scheme: B=Bookwork, C=New computed example, A=Analysis of new circuit, D=design of new circuit

1. (a) Nodal equation at X gives:

$$\frac{X-19}{3}+3+\frac{X}{2}=\frac{5X-38+18}{6}=0 \text{ from which } X=4\,\text{V}.$$ [5A]

(b) If we set the current source to zero, we are left with a potential divider and so

$$V_{AB}=25\times\frac{3}{5}=15$$

Now setting the voltage source to zero, we have a current divider and $\frac{2}{5}$ of the $5\,\text{mA}$ total current will flow through the 3k resistor. Thus the voltage will be $V_{AB}=-6\,\text{V}$. [5A]

By superposition, the total voltage is therefore $15-6=9\,\text{V}$.

(c) The Thévenin resistance (obtained by setting the current source to zero) is $3 \| 9 = 2.25\,\text{k}\Omega$.

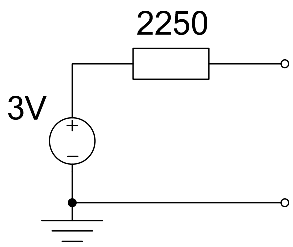
To obtain the open-circuit voltage, there are several possible approaches:

(i) we note that we have a current divider with resistances of 2k and 10k. Therefore $\frac{1}{6}$ of the total current will flow through the 7k+3k combination giving an open circuit voltage of 3 V.

(ii) An alternative approach is to first convert the leftmost two components into their own mini Thévenin equivalent (12V and 2k), the open circuit voltage is now easily derived as $12 \times \frac{3\text{k}}{12\text{k}}$.

(iii) If E is the voltage across the 2k resistor, you can use nodal analysis to write down: $\frac{E}{2} - 6 + \frac{E-C}{7} = 0$ and $\frac{C-E}{7} + \frac{C}{3} = 0$ which you can solve to get $C = 3$. Alternatively, you can write a single equation for E as: $\frac{E}{2} - 6 + \frac{E}{10} = 0 \Rightarrow E = 10$ and then get $C = 3$ from a potential divider. [5A]

The Thévenin equivalent is therefore:



Comment: Yet another way to calculate the Thévenin voltage is to multiply the Thévenin resistance by the short-circuit current; however almost everyone who tried this method get it wrong. Many people who used method (iii) above confused E and C and ended up with C=10.

(d) (i) When $X = +5$, the diode is on and so $Y = X - 0.7 = 4.3\,\text{V}$.

As a check, the current through the 7k resistor is 0.1 mA, while the current through the 3k resistor is $\frac{4.3}{3} = 1.43\,\text{mA}$. This implies that there is indeed a forward diode current of 1.33 mA. [2A]

(ii) When $X = -5$, the diode is off and we have a potential divider: $Y = \frac{3}{10}X = -1.5\,\text{V}$. [3A]

(e) The circuit is a potential divider: $\frac{Y}{X} = \frac{Z_{RC}}{Z_L + Z_{RC}} = \frac{1}{1 + Z_L / Z_{RC}}$

$$Z_{RC} = \frac{-12j}{6-2j} = 0.6 - 1.8j \quad \Rightarrow \quad \frac{Y}{X} = \frac{-12j}{-12j + 10j(6-2j)} = \frac{-12j}{20+48j}$$

$$Y = X \times \frac{-3j}{5+12j} = 2j \times \frac{-3j}{5+12j} = \frac{30-72j}{169} = 0.1775 - 0.426j = \frac{6}{13}\angle -67^\circ$$ [3A]

The corresponding waveform is $y(t) = 0.1775\cos(\omega t) + 0.426\sin(\omega t)$. [2A]

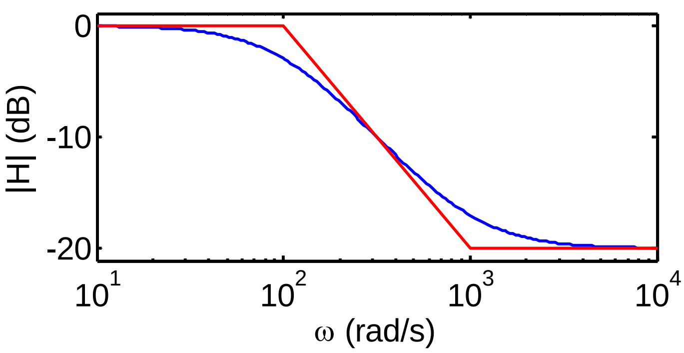
Comment: Many candidates made arithmetic slips doing the complex arithmetic presumably because they did not know how to use the calculator's complex arithmetic mode and its memories for temporary storage. Quite a few did not know the direct mapping from phasor to Acos(wt)+Bsin(wt) but instead converted the phasor to polar form and then used trigonometry to convert it back again (often with errors).

(f) This circuit is a potential divider, so its transfer function is

$$\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{10R + \frac{1}{j\omega C}} = \frac{1 + j\omega RC}{1 + 10j\omega RC}$$ [3A]

The low and high frequency gains are 1 and 0.1 respectively (0 and $-20\,\text{dB}$).

If $RC = 0.001$, the corner frequencies are at $\frac{0.1}{RC} = 100\,\text{rad/s}$ and $\frac{1}{RC} = 1000\,\text{rad/s}$. [2A]



(g) We first calculate the current phasor:

$$\tilde{I} = \frac{\tilde{X}}{2+10j} = \frac{25-125j}{13} = 1.92 - 9.62j = 9.81\angle -79°$$ [2C]

Power absorbed by source is [1C]

$$-100\tilde{I}^* = \frac{-2500-12500j}{13} = -192-962j \text{ VA} = 981\angle\text{-}101°.$$

The negative sign arises from the passive sign convention. Thus the source is generating average power and generating VARs.

Power absorbed by resistor is $S_R = \left|\tilde{I}\right|^2 R = 9.81^2 \times 2 = 192 + 0j$ W [2C]

Power absorbed by inductor is $S_L = \left|\tilde{I}\right|^2 Z = 9.81^2 \times 10j = 962j$ VA.

The last two calculations can more easily be deduced from the first without doing any additional calculations since we know (from Tellegen's theorem) that the total complex power must equal zero.

Comments: Many candidates omitted the modulus signs and wrongly took $S_R = \tilde{I}^2 R$.

(h) (i) The steady state is $Y = X$ and so the steady states are 5 and 10 V respectively. [2C]

(ii) The capacitor voltage cannot change instantly, and so $y(0+) = y(0-) = 5$. The transient amplitude is therefore -5 and the full equation is [3C]

$$y(t) = 10 - 5e^{-\frac{t}{RC}}.$$

2. (a) The easiest way is to spot that this is a potential divider followed by a non-inverting amplifier. Its gain is therefore [6A]

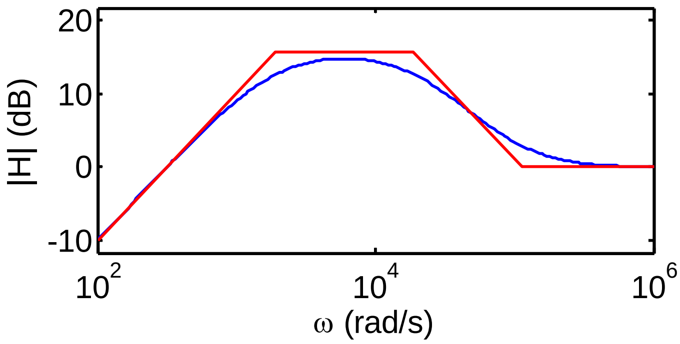
$$\frac{Y}{X} = \frac{Z_2}{Z_1 + Z_2} \times \left(1 + \frac{Z_4}{Z_3}\right) = \frac{Z_2(Z_3 + Z_4)}{Z_3(Z_1 + Z_2)}$$

(b) We now have $Z_4 = \dfrac{R \times \frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \dfrac{R}{1 + j\omega RC}$. From the answer to part (a), we get

$$\frac{Y}{X} = \frac{10R\left(0.2R + \frac{R}{1 + j\omega RC}\right)}{0.2R\left(\frac{1}{j\omega C} + 10R\right)} = \frac{10 + \frac{50}{1 + j\omega RC}}{10 + \frac{1}{j\omega RC}} = \frac{60\, j\omega RC\left(1 + \frac{1}{6} j\omega RC\right)}{(1 + j\omega RC)(1 + 10\, j\omega RC)}$$ [6A]

(c) The low and high frequency asymptotes are respectively $\dfrac{10\, j\omega RC \times 1.2}{0.2} = 60\, j\omega RC$ and $\dfrac{10 \times 0.2}{0.2 \times 10} = 1$. [4C]

The numerator corner frequency is $\dfrac{1.2}{0.2RC} = \dfrac{6}{RC}$. The denominator corner frequencies are $\dfrac{1}{RC}$ and $\dfrac{0.1}{RC}$. Hence the magnitude response looks like this: [3C]



[4C]

The high frequency asymptote has a gain of 1 (as is clear from the circuit in any case). Since the two highest corner frequencies differ by a ratio of 6, it follows that the gain of the central flat portion of the approximate magnitude response is also $\left|\dfrac{Y}{X}\right| = 6 = 15.6\,\text{dB}$. Alternatively, you can substitute $\omega = \dfrac{0.1}{RC}$ into the low frequency asymptote expression: $60\, j\omega RC$. [3C]

(d) The lowest corner frequency is at $\dfrac{0.1}{RC} = 2\pi \times 300\,\text{Hz}$. So we need

$$C = \frac{0.1}{2\pi \times 300 \times 10\text{k}} = 5.3\text{ nF}$$ [4C]

Comment: Most were able to do (a) OK although some required a large amount of algebra. Widespread poor algebra in part (b) led to many dimensionally inconsistent expressions and many pages of working. Also many did not give the transfer function in factorized form. In part (b) very many people multiplied out the denominator and then, a few lines of working later, tried to factorize it (often incorrectly). Many candidates failed to spot the connection between parts (a) and (b) and started again from scratch.

3. (a) $\tilde{I} = \frac{\tilde{V}}{Z} = \frac{240}{7.2+9.6j} = 12-16j = 20\angle -53° \quad \Rightarrow \quad |\tilde{I}|^2 = 400$ [4A]

Hence average power absorbed by the line is $|\tilde{I}|^2 R_{line} = 400 \times 1.2 = 480\,\text{W}$ and the average power absorbed by the load is $|\tilde{I}|^2 R_{load} = 400 \times 6 = 2.4\,\text{kW}$. [4C] [4C]

(b) The current lags the voltage and the phase angle is $\tan^{-1}\left(\frac{16}{12}\right) = 0.93\,\text{rad} = 53°$. [4C]

(c) The load admittance is $\frac{1}{6+8j} = 60-80j\,\text{mS}$. [3A]

Hence we need an admittance of $j\omega C = 80j\,\text{mS} \quad \Rightarrow \quad C = \frac{0.08}{2\pi \times 50} = 255\,\mu\text{F}$. [3A]

The capacitor impedance is $\frac{1}{0.08j} = -12.5j\,\Omega$.

(d) With the capacitor in place, the load admittance becomes 60 mS which corresponds to a resistance of $16.7\,\Omega$. The current is therefore now [4A]

$$\tilde{I} = \frac{\tilde{V}}{Z} = \frac{240}{17.9+1.6j} = \frac{2412-216j}{181} = 13.3-1.2j = 13.4\angle -5.1° \quad \Rightarrow \quad |\tilde{I}|^2 = 179$$

The power dissipated in the line has now decreased to $|\tilde{I}|^2 \times 1.2 = 214.8\,\text{W}$ [4C]

4. (a) To determine the time constant we short circuit the input voltage source at X and find the Thévenin resistance seen by the capacitor. Since the $4R$ and R resistors are in parallel, this resistance is $0.8R$. Hence the time constant is $0.8RC$. [4C]

Alternatively, you can work out the transfer function: $\frac{Y}{X} = \frac{1+4j\omega RC}{5+4j\omega RC}$ and the denominator corner frequency: $\omega_D = \frac{5}{4RC}$. Then $\tau = \frac{1}{\omega_D} = 0.8RC$.

(b) (i) The steady state gain for DC is $\frac{Y}{X} = \frac{R}{5R} = 0.2$ so the steady state outputs are $Y = 0$ and $Y = 2$. [5A]

(ii) The voltage across the capacitor cannot change instantly and so $y(0+) = 10$. Hence the transient amplitude is 8 and $y(t) = 2 + 8e^{\frac{-t}{0.8RC}}$. [5A]

(c) (i) We first note that $4R \,||\, C = \frac{4R \times \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{4R}{1+4j\omega RC}$.

Now, viewing the circuit as a potential divider, the transfer function is $\frac{Y}{X} = \frac{R}{R + \frac{4R}{1+4j\omega RC}} = \frac{1+4j\omega RC}{5+4j\omega RC}$ giving low and high frequency asymptotes of 0.2 and 1. [5A]

(ii) For $\omega = \frac{2}{RC}$, $\frac{Y}{X} = \frac{1+8j}{5+8j} = \frac{69+32j}{89} = 0.775 + 0.36j = 0.855\angle 24.9°$. [6A]

Since, as a phasor, $X = -j$, $Y = 0.36 - 0.775j = 0.855\angle -65.1°$ so the steady state component is $y(t) = 0.36\cos(\omega t) + 0.775\sin(\omega t)$.

(iii) At $t = 0+$, $x(0+) = 0$ and so, since the capacitor voltage cannot change instantly, $y(0+) = 0$. Therefore, the transient amplitude is -0.36 and the full expression is $y(t) = 0.36\cos(\omega t) + 0.775\sin(\omega t) - 0.36e^{\frac{-t}{0.8RC}}$. [5A]

Comments: In part (a) candidates lost marks because they did not realize that the word "show" in the question required them to explain why the resistors were in parallel. In part (b.ii), many stated correctly that the capacitor voltage could not change instantly, but actually assumed that $y(t)$ could not change instantly (which is untrue). In part (c.ii) quite a number of people confused "steady state" with "DC" and assumed that $\frac{Y}{X} = 0.2$ applied to a sine wave input. Several used a formula that $y(t) = y(0+) + \left(y_{SS}(t) - y(0+)\right)\left(1 - e^{-t/\tau}\right)$ instead of the one given in lectures: $y(t) = y_{SS}(t) + \left(y(0+) - y_{SS}(0+)\right)e^{-t/\tau}$; the two formulae are equivalent for a DC steady state but the first formula is wrong if $y_{SS}(t)$ varies with time. The transient amplitude should never be time-varying.