Paper Number(s): **EE1-1**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE
UNIVERSITY OF LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2011

EEE/ISE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Friday, 3 June 10:00 am

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Time allowed: 2:00 hours

Examiners responsible:

First Marker(s): D.M. Brookes

Second Marker(s): P. Georgiou

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square phasor voltage by $\tilde{X} = \frac{X}{\sqrt{2}}$.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.

1. (a) Using nodal analysis calculate the voltages at nodes X and Y in *Figure 1.1*. [5]

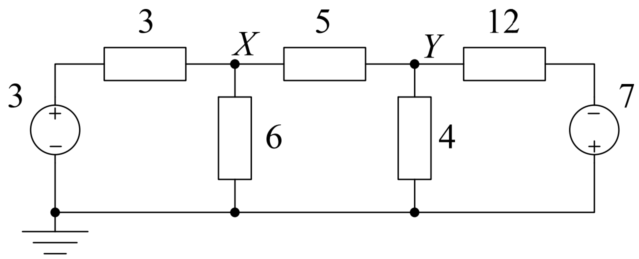


Figure 1.1

(b) Use the principle of superposition to find the voltage V in *Figure 1.2*. [5]

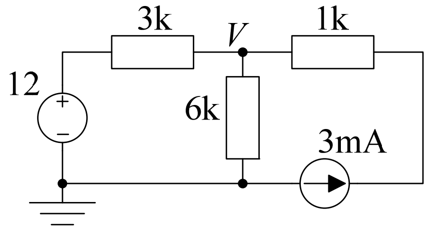


Figure 1.2

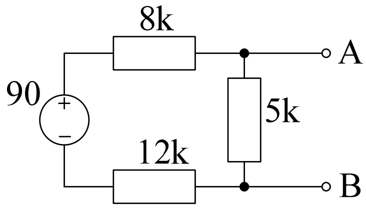


Figure 1.3

(c) Draw the Thévenin equivalent circuit of the network in *Figure 1.3* and find the values of its components. [5]

(d) Assuming the opamp in the circuit of *Figure 1.4* is ideal, give an expression for Z in terms of X. [5]

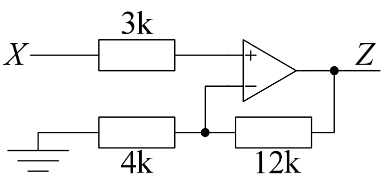


Figure 1.4

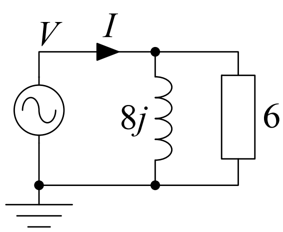


Figure 1.5

(e) (i) The phasor representing the voltage at V in *Figure 1.5* has the value $24j$. Determine the phasor current I in the form $a + jb$. [2]

(ii) Determine the complex impedance of the parallel L-R combination in the form $r\angle\theta$. [2]

(iii) If $\omega = 500$ rad/s, calculate the value of the inductance in Henries. [1]

(f) (i) Show that the frequency response of the circuit shown in Figure 1.6 is [1]

$$\frac{Y}{X} = \frac{1}{j\omega RC + 1}$$

(ii) Give expressions for the low and high frequency asymptotes of the response. [1]

(iii) Draw separate graphs showing straight-line approximations to the magnitude and phase responses of the circuit. Indicate on your graphs the corner frequency values and the values of any horizontal portions of the responses. [3]

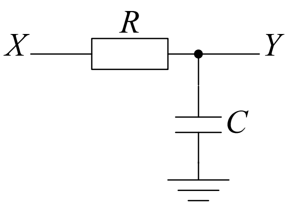


Figure 1.6

(g) (i) Determine the angular frequency, ω_0, at which the impedances of the inductor and capacitor in the circuit of *Figure 1.7* have the same magnitude. [2]

(ii) Determine the value of the phasor X at the frequency ω_0 if the phasor V has the value 10. [3]

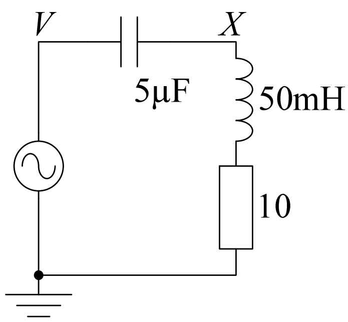


Figure 1.7

(h) In *Figure 1.8*, the voltage at X is $x(t) = 5\sin\omega t$. Sketch a graph showing the waveform at Y. Indicate on your graph the maximum and minimum values taken. Assume that the diode has a forward voltage drop of 0.7 V and is otherwise ideal. [5]

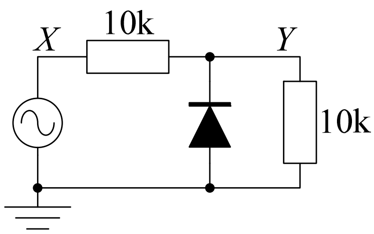


Figure 1.8

2. (a) Assuming that the op-amp in the circuit of *Figure 2.1* is ideal, give an expression for the gain $\frac{Y}{X}$. State clearly any assumptions you make. [4]

(b) Determine the transfer function, $\frac{Y}{X}(j\omega)$ of the circuit shown in *Figure 2.2*. Give expressions for its low and high frequency asymptotes and for its corner frequencies. [10]

(c) Using logarithmic axes, draw a dimensioned sketch of the straight-line approximation to the magnitude response, $\left|\frac{Y}{X}(j\omega)\right|$, when $C = 10$ n, $R_1 = 10$ k and $R_2 = 25$ k. Indicate on your sketch, the values of the corner frequencies in Hz and the gain of any horizontal portions of the response. [6]

(d) Explain how the response would be changed if the two resistors were interchanged. [4]

(e) With $C = 10$ n, select values for R_1 and R_2 so that the corner frequencies are at 500 Hz and 5 kHz and the gain of the horizontal portion of the transfer function has unit magnitude. [6]

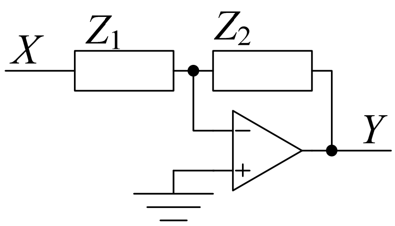


Figure 2.1

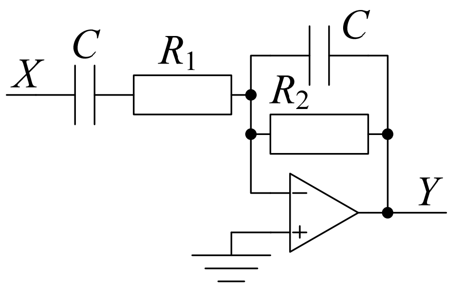


Figure 2.2

3. In the circuit of *Figure 3.1* the two resistors have values Y and R, the complex impedance of the inductor is jX and the phasor voltage of the source is V at an angular frequency $\omega = 500$ rad/s.

(a) Give an expression for the average power dissipation of resistor R in terms of V, R, X and Y. [8]

(b) Prove that the value of R that maximizes its average power dissipation is given by [6]

$$R = \sqrt{X^2 + Y^2}.$$

(c) If $V = -10j$, $X = 500$, $Y = 10$ and $R = 50$ determine the complex power absorbed by each of the components and the phasor voltage W in the form $a + jb$ where a and b are decimal numbers. [8]

(d) Now suppose that the component values are the same as in part (c), but the waveform V is now given by

$$v(t) = \begin{cases} 0 & \text{for } t < 0 \\ 10 \sin 500t & \text{for } t \geq 0 \end{cases}$$

as shown in *Figure 3.2*.

Determine an expression for the waveform $w(t)$ for $t \geq 0$. Calculate the numerical values of all quantities in the expression. [8]

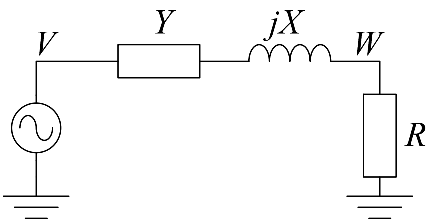


Figure 3.1

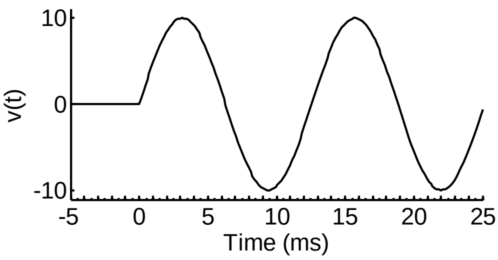


Figure 3.2

2011 E1.1: Analysis of Circuits - Solutions

Key to letters on mark scheme: B=Bookwork, C=New computed example, A=Analysis of new circuit, D=design of new circuit

1. (a) Nodal equation at X gives $\frac{X-3}{3}+\frac{X}{6}+\frac{X-Y}{5}=0$ from which $21X-6Y=30$. [This simplifies to $7X-2Y=10$.] [2A]

Nodal equation at Y gives $\frac{Y-X}{5}+\frac{Y}{4}+\frac{Y+7}{12}=0$ from which $12X-32Y=35$. [2A]

Taking 4 times the first equation minus 7 times the second gives $200Y=-125$ from which $Y=-0.625$. Substituting this into the second equation gives $12X=15$ from which $X=1.25$. [1A]

The commonest mistake was the wrong sign for the 7V source. Many use a lot of algebra to solve the simultaneous equations (the supplied calculator will do it directly) often including sign-errors or other algebraic mistakes.

(b) Setting the current source to zero (open circuit) gives a potential divider with $V=12\times\frac{6}{9}=8$. Setting the voltage source to zero (short circuit) gives 3k and 6k resistors in parallel which are equivalent to 2k. Hence the voltage due to the current source is $2\times 3=6$. Combining these gives $V=8+6=14$. [5A]

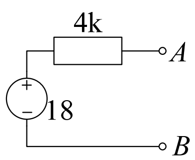
The commonest mistake was the wrong sign for the effect of the 3mA source.

(c) The Thévenin resistance (obtained by setting the voltage source to zero) is $5||(8+12)=5||20=4\text{ k}\Omega$. [2A]

The open circuit voltage is just the voltage across the 5k resistor. The circuit is a potential divider so this is $V_{Th}=90\times\frac{5}{8+5+12}=18\text{ V}$. [2A]

Several took the Thévenin voltage to be the combined voltage across the 12k+5k resistors instead of just across the 5k resistor. Several used nodal analysis to work out VA and VB; this is OK but wasteful of effort. If, like some, you don't fix a ground reference node, the nodal analysis method is doomed to failure (the best point to choose as ground is B rather than the negative end of the voltage source).

The Thévenin equivalent is therefore:



[1A]

(d) There is no current through the 3k resistor, so V+ will equal X. The amplifier is a non-inverting amplifier, so $Z = \left(1 + \frac{12}{4}\right)X = 4X$. [5A]

Incidentally, the inclusion of the 3k resistor is not merely perverse. Making the source impedance the same at the two op-amp input terminals largely eliminates errors due to non-zero op-amp input currents (e.g. at high frequencies). Quite a few people gave this and other answers in an irritatingly unevaluated form, e.g. $Z = \left(1 + \frac{12}{4}\right)X$.

(e) (i) $I = \frac{24j}{8j} + \frac{24j}{6} = 3 + 4j$. [2A]

It is easier to work out the total current by summing two the individual currents through the components than by calculating the parallel impedance.

(ii) $Z = \frac{6\times 8j}{6+8j} = \frac{96+72j}{25} = 3.84 + 2.88j = 4.8\angle 0.644 = 4.8\angle 36.9°$ [2A]

(iii) $j\omega L = 8j$ so $L = \frac{8}{\omega} = \frac{8}{500} = 16$ mH [1A]

(f) (i) This circuit is a potential divider, so its transfer function is [1A]

$$\frac{Y}{X} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$$

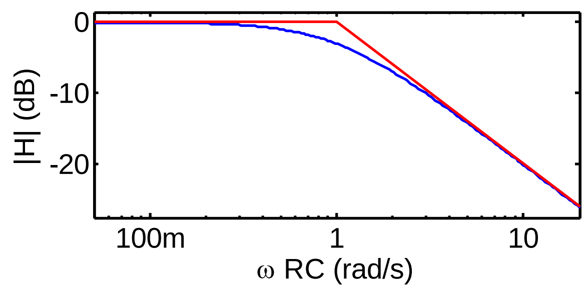
(ii) The LF asymptote is 1 and the HF asymptote is $\frac{1}{j\omega RC}$. [1A]

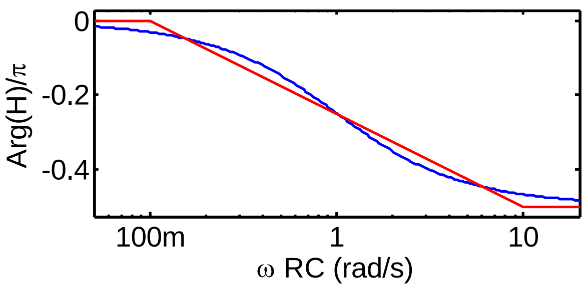
The asymptote should include the j in the denominator since you are not asked for the asymptote of the magnitude response. Many people gave the HF asymptote as 0 (since this is its value at $\omega = \infty$) but this is incorrect.

(iii) The magnitude response corner frequency is at $\omega = \frac{1}{RC}$ with the LF asymptote having a gain of 0 dB.

The phase corner frequencies are at $\omega = \frac{0.1}{RC}$ and $\omega = \frac{10}{RC}$. The horizontal phase asymptotes are at 0 and $-\frac{\pi}{2}$ respectively. [3A]

Quite often the phase plot did not match the LF and HF phases (0 and $-\frac{\pi}{2}$) that can be deduced from the LF and HF asymptotes in part (ii). The question explicitly asked for the straight line approximation (red on the graphs below) rather than a vague sketch of the true (blue) curve; not everyone gave this. Several people omitted the phase plot entirely.





(g) (i) We require $\left|\frac{1}{j\omega_0 C}\right| = |j\omega_0 L|$ from which we get $\omega_0 = \sqrt{\frac{1}{LC}} = 2000$ rad/s. [2A]

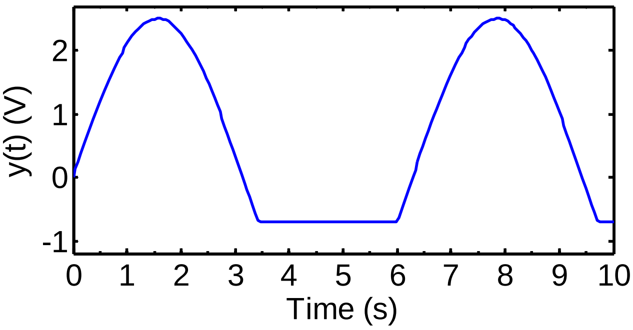
Several wrongly gave the units as Hz. Others missed out the modulus signs and got $\omega_0 = 2000j$ which doesn't really make sense.

(ii) $\frac{1}{\omega_0 C} = \omega_0 L = 100$ so $\frac{X}{V} = \frac{10+100j}{10+100j-100j} = \frac{10+100j}{10} = 1 + 10j$.
Hence $X = 10 + 100j$. [3A]

Several people took $Z_C = +100j$ missing the whole point of the circuit which is that at resonance Z_C and Z_L cancel out.

(h) When the diode is off, $y(t) = 0.5x(t)$, however when the diode is on, $y(t) = -0.7$. Thus $y(t) = \max(0.5x(t), -0.7)$ which gives the graph below. The maximum value of $y(t)$ is 2.5 and the minimum is -0.7. [5A]

A surprising number of errors. Some tried to use phasors (no good in a non-linear circuit) and some of these concluded that the diode was permanently off by comparing the phasor value of X to +0.7 or -0.7. Others treated the diode as being in series with the rightmost resistor. Many people got the correct voltage across the diode in forward bias (-0.7 V) but then said Y was something completely different.



2. (a) The gain is $\frac{Y}{X} = -\frac{Z_2}{Z_1}$. We assume that there is no current into the input terminals of the op-amp and that the op-amp gain is infinite: this implies that negative feedback will result in the input terminals having the same voltage. [4A]

Several failed to mention the zero input current assumption.

(b) Refering to the previous part, $Z_1 = R_1 + \frac{1}{j\omega C} = \frac{j\omega R_1 C+1}{j\omega C}$ and $Z_2 = \frac{R_2 \times \frac{1}{j\omega C}}{R_2 + \frac{1}{j\omega C}} = \frac{R_2}{j\omega R_2 C+1}$.

Quite a few people did the nodal analysis from scratch without using the result of part (a).

Substituting these expressions into the gain equation from part (a) gives [6A]

$$\frac{Y}{X} = -\frac{R_2}{j\omega R_2 C + 1} \times \frac{j\omega C}{j\omega R_1 C + 1} = -\frac{j\omega R_2 C}{(j\omega R_1 C + 1)(j\omega R_2 C + 1)}.$$

Several people multiplied out the denominator (and in some cases tried to factorize it later); it is usually better to keep it factorized. Some failed to factorize it and thought it was a quadratic resonance.

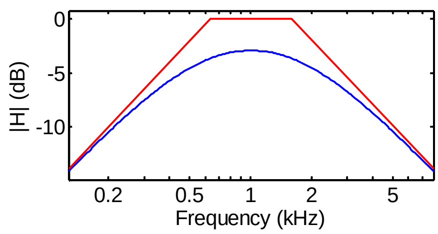
Note that the gain is a voltage ratio and is therefore dimensionless. This means that you can write the transfer function and the asymptotes so that all terms are dimensionless quantities such as $j\omega RC$ (as is done above). In many cases algebraic errors led to dimensionally incorrect or inconsistent expressions; these are are easily identifiable and always wrong.

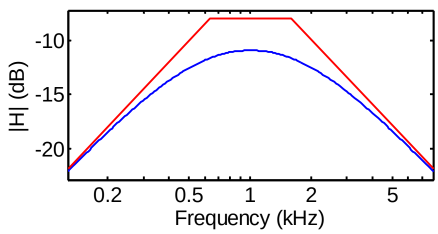
The LF asymptote is $-j\omega R_2 C$ and the high frequency asymptote is $\frac{-1}{j\omega R_1 C}$.

Several wrongly said the LF asymptote was 0. An asymptote is a straight line that the response tends to for small or large ω. Only if the asymptote is a multiple of $(j\omega)^0$ (not true in this case) is it equal to $H(j0)$ or $H(j\infty)$. Many people omitted the j from the asymptotes; this is correct for the "magnitude response asymptote" but not what was asked in the question.

The corner frequencies are $\frac{1}{R_1 C}$ and $\frac{1}{R_2 C}$. [4A]

(c) With the values given, the corner frequencies are 4k and 10k rad/s = 637 Hz and 1592 Hz. Between these frequencies, the straight line approximation gives a gain of 0 dB (see left hand plot). [6A]





Quite often the LF and HF slopes did not match the exponents of $j\omega$ in the LF and HF asymptote expressions calculated in part (b). Several thought that the slope of $A(j\omega)^m$ depended on A as well as m; an extreme version was thinking that negative A made the slopes go up instead of down. A less extreme version was to have different slopes on the two sides.

Quite a few thought that a gain of 0 and a gain of 0dB were the same thing.

(d) We now have $R_1 = 25\text{k}$ and $R_2 = 10\text{k}$, The corner frequencies remain the same but the lowest-valued corner frequency is now $\frac{1}{R_1 C}$. If we calculate the value of the LF asymptote at this frequency, we find that the mid-band gain has been reduced to $\frac{R_2}{R_1} = 0.4 = -8$ dB (see right-hand plot). The response is identical but shifted down. Another way of seeing this is that in the expression given in part (b) above, the denominator is unchanged but the numerator is multiplied by $\frac{10}{25}$. [4A]

(e) We need $[R_1, R_2] = \frac{1}{2\pi f C} = [3.18\text{k}, 31.8\text{k}]$. Note that they must be in this order, or else the mid-band gain is -20 dB. [6D]

Many people omitted the 2π factor.

3. (a) The current is $=\frac{V}{R+Y+jX}$. The power dissipated in R is $\left|\tilde{I}\right|^2 R=\frac{1}{2}|I|^2 R$. Substituting for I gives [8A]

$$P=\frac{1}{2}\times\frac{|\mathrm{V}|^2}{(\mathrm{R}+\mathrm{Y})^2+\mathrm{X}^2}\times R=\frac{\frac{1}{2}|V|^2 R}{(R+Y)^2+X^2}$$

Many people omitted the factor of $\frac{1}{2}$.

When finding $|I|^2=\left|\frac{V}{R+Y+jX}\right|^2$ many people first rationalized the fraction by multiplying the numerator and denominator by $R+Y-jX$. This is a really <u>bad idea</u> because it turns it into a much more complicated expression and makes part (b) especially hard. Much better is to notice that $\left|\frac{V}{R+Y+jX}\right|^2=\left(\frac{V}{R+Y+jX}\right)\left(\frac{V}{R+Y+jX}\right)^*=\frac{|V|^2}{|R+Y+jX|^2}=\frac{|V|^2}{(R+Y)^2+X^2}$.

(b) We want to find the value of R that makes $\frac{dP}{dR}=0$. We can ignore the constant factor and need only consider the numerator of $\frac{dP}{dR}$. This gives (from the quotient rule):

$$\frac{dP}{dR}\propto\{(R+Y)^2+X^2\}\times 1-R\times\{2(R+Y)\}=-R^2+Y^2+X^2$$

Setting this to zero gives $R=\sqrt{X^2+Y^2}$. [6A]

Although not requested, this gives $P=\frac{|V|^2}{4(R+Y)}$.

Since there is only one stationary point, $P\geq 0$ always and $R=0\Rightarrow \mathrm{P}=0$, then it must be a power maximum. Alternatively, you can differentiate again and show that the second derivative is negative.

Many people apparently did not know how to find the maximum of a function and skipped this bit entirely. Others just stated the answer without any algebra: "proof by assertion". Others just substituted the value given for R into their power expression. If the power calculated in part (a) was complex-valued then there is no such thing as a maximum.

A more obscure, purely algebraic, method is to minimize the reciprocal as follows:

$$\frac{1}{P}\propto\frac{(R+Y)^2+X^2}{R}=\frac{R^2+2RY+Y^2+X^2}{R}=R+2Y+\frac{X^2+Y^2}{R}$$
$$=\left(\frac{\sqrt{X^2+Y^2}}{\sqrt{R}}-\sqrt{R}\right)^2+2\sqrt{X^2+Y^2}+2Y$$

The first term is the only one involving R and it can never be negative and so the expression is minimized by making it equal to zero which happens when $R=\sqrt{X^2+Y^2}$.

One person used essentially this method.

(c) We have $I=\frac{V}{R+Y+jX}=\frac{-25-3j}{1268}=(-19.7-2.37j)\mathrm{mA}=19.9\angle-173°\ \mathrm{mA}$.

Hence $W = IR = \frac{-625-75j}{634} = -0.986 - 0.118j = 0.993\angle -173°$ [2A]

The complex power absorbed by a component with impedance Z is $|\tilde{I}|^2 Z$.

We can calculate $|\tilde{I}|^2 = \frac{1}{2}|I|^2 = \frac{1}{5072} = 1.97 \times 10^{-4}$.

Therefore the complex power absorbed by each component is:

- Y: $\frac{5}{2536} = 1.98$ mW
- jX: $\frac{125j}{1268} = 98.6j$ mVA
- R: $\frac{25}{2536} = 9.86$ mW
- Source: $-\tilde{V}\tilde{I}^* = \frac{-15-125j}{1268} = -11.8 - 98.6j$ mVA

As expected, the total power sums to zero. [6A]

Other expressions like $S = \tilde{V}\tilde{I}^* = \frac{|\tilde{V}|^2}{\tilde{Z}^*}$ are also valid but computationally more effort for the components. Many used I instead of $\tilde{I}$ and so got answers that were twice as big as the correct ones. Several forgot to calculate W at all. Quite a few gave their results to only 1 significant digit which is rather low precision.

(d) We have already calculated the steady state phasor $W = -0.986 - 0.118j$. This implies that the waveform $w(t) = -0.986\cos\omega t + 0.118\sin\omega t + Ae^{-\frac{t}{\tau}}$.

We can see that $\tau = \frac{L}{R+Y} = \frac{X}{\omega(R+Y)} = \frac{1}{60} = 16.7$ ms.

We know that $w(t)$ cannot have a discontinuity at $t = 0$ both because V is continuous and also because the current through the inductor cannot change instantly and so must be zero at time $t = 0+$.

It follows that $A = 0.986$. [8A]

Several people thought "steady state" meant the same as "DC" and used the DC gain of $\frac{50}{60}$. However "steady state" in this case is the phasor W that was calculated in the part (c). Many people got the sign of A incorrect. Many people calculated a transient amplitude that was either complex or else depended on t.