IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2013

EEE/EIE PART I: MEng, Beng and ACGI

ANALYSIS OF CIRCUITS

Tuesday, 28 May 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes
Second Marker(s) : P. Georgiou

ANALYSIS OF CIRCUITS

**** Questions and Solutions 2013 ****

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.

*** Questions and Solutions 2013 ***

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1.

[5]

The unlabelled node at the -ve end of the 6V *source has voltage* $Y-6$ *and forms a supernode with node* Y*. KCL at* X *gives*

$$\begin{aligned}\frac{X-23}{2}+\frac{X-(Y-6)}{4}+\frac{X-Y}{2} &= 0\\ \Rightarrow 5X-3Y &= 40\end{aligned}$$

KCL at Y *gives*

$$\begin{aligned}\frac{Y-X}{2}+\frac{(Y-6)-0}{3}+\frac{(Y-6)-X}{4} &= 0\\ \Rightarrow -9X+13Y &= 42\end{aligned}$$

Combining these gives $65Y-27Y=210+360 \quad \Rightarrow \quad Y=\frac{570}{38}=15$

form which $5X=40+45=85 \quad \Rightarrow \quad X=\frac{85}{5}=17$

A few people introduced another variable, Z, for the voltage at the unlabelled node instead of using $Y-6$; this is not incorrect but complicates the algebra. Some got sign errors when expanding $-(Y-6)$. Most people got the equations right although a few failed to handle the supernode correctly. Several omitted the term $\frac{(Y-6)-X}{4}$ and/or $\frac{(Y-6)-0}{3}$ from the second equation. Several people assumed that $Y=6$. Quite a few made algebra errors when solving the simultaneous equations, often involving minus signs. I recommend writing equations consistently in the form $aX+bY=c$; those who sometimes wrote them in the other order (e.g. $bY+aX=c$) frequently made mistakes. This is especially true if you use the calculator's built-in simultaneous equation solver.

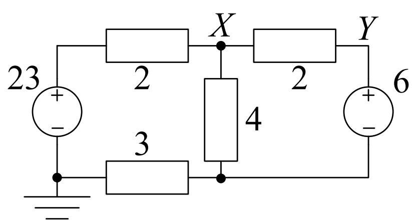

Figure 1.1

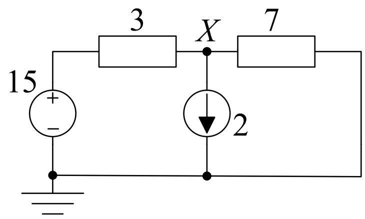

Figure 1.2

b) Use the principle of superposition to find the voltage at node X in Figure 1.2.

[5]

If we open circuit the current source, we get a potential divider with $X_1 = 15\times\frac{7}{10}=10.5$.

If we short circuit the voltage source, the 3Ω *and* 7Ω *resistors are an parallel and combine to give* $\frac{3\times7}{3+7}=2.1\Omega$*. Thus the voltage due to the current source is* $X_2=-2\times2.1=-4.2$.

Combining these gives $X = X_1 + X_2 = 10.5 - 4.2 = 6.3\,\mathrm{V}$.

In most questions, you can use any solution method you like. However in this question, you are told explicitly to use superposition; so if you solve it using nodal analysis, you get zero marks.

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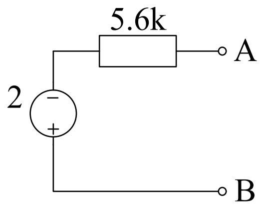
c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the value of its components. [5]

We can find the open circuit voltage by ignoring the 2k *resistor and treating the other two as a potential divider. This gives an open circuit voltage of* $V_{AB} = -8 + \frac{6}{6+9} \times (7 - (-8)) = -8 + 6 = -2$.

For the Thévenin resistance, we short circuit the sources to give $\frac{6\times 9}{6+9} + 2 = 3.6 + 2 = 5.6\,\mathrm{k}$.

So the complete Thévenin equivalent is:



Some calculated the component values but lost marks because they did not draw the circuit as required by the question. Several people drew the circuit wrongly by adding a ground connection, omitting the terminals A and B or sometimes even short circuiting A to B.

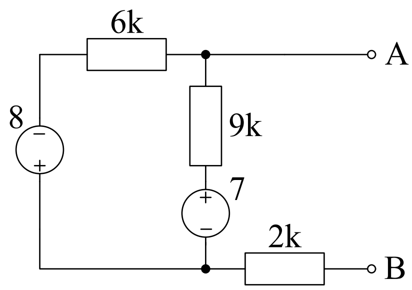


Figure 1.3

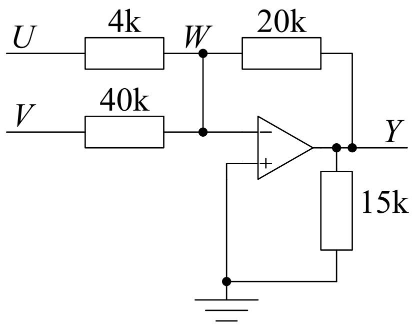


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

This is an inverting op-amp circuit and so we can write down $Y = \frac{-20}{4}U + \frac{-20}{40}V = -5U - 0.5V$.

Alternatively, assuming node W *is a ground and applying KCL at node* W *gives* $\frac{0-U}{4} + \frac{0-V}{40} + \frac{0-Y}{20} = 0$ *from which* $Y = -5U - 0.5V$.

Note that the $15\,\mathrm{k}$ *resistor has no effect on the answer.*

Mostly done correctly. The 15k resistor caused some confusion; several people thought that it would affect the voltage at the +ve op-amp input (even though this is very clearly connected to ground in the circuit). Some did KCL at the +ve op-amp input and ignored the ground connection to get the wrong equation $\frac{Y-V_+}{15}=0$. Others added an extra term, $\frac{Y-0}{15}$ onto the KCL at node W.

---

e) Determine the gain $\frac{Y}{X}$ for the block diagram shown in Figure 1.5. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors whose inputs have the signs indicated on the diagram and whose outputs are V and W respectively. [5]

From the block diagram we can write $Y = GW = G(FV+Y) = G(F(X-Y)+Y)$.

Hence $Y = GFX - GFY + GY$ from which $Y(1+GF-G) = GFX$ and so $\frac{Y}{X} = \frac{FG}{1-G+FG}$.

Mostly done correctly. Quite a few people got the adder functionality wrong when forming the equations and, for example, wrote $W = FV + X$ instead of $W = FV + Y$. One or two applied KCL to the adder nodes to form entirely incorrect equations. Several people had the blocks with inputs on the right and wrote $W = GY$ instead of $Y = GW$ even though convention (and the question explicitly) has signals flowing from left to right.

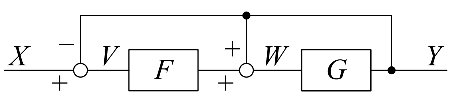


Figure 1.5

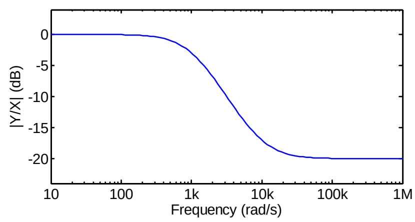
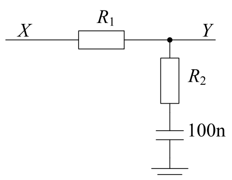


Figure 1.6

f) Using a single 100 nF capacitor and appropriate resistors, design a network with input X and output Y whose transfer function $\frac{Y}{X}(j\omega)$ is shown in Fig. 1.6 with corner frequencies at 10^3 and 10^4 rad/s. Give the values of all components used. [5]

The network needs a gain of 1 at low frequencies falling to a gain of 0.1 (-20 dB) at high frequencies. Therefore we make it a potential divider



The transfer function of this network is $\frac{R_2+\frac{1}{j\omega C}}{R_1+R_2+\frac{1}{j\omega C}} = \frac{j\omega R_2 C+1}{j\omega(R_1+R_2)C+1}$. The LF and HF asymptotes are 1 and $\frac{R_2}{R_1+R_2}$ respectively, from which $R_1 = 9R_2$. The *numerator corner frequency is at* 10^4 *rad/s (since the gradient increases by* 1 *at this frequency) and so* $\frac{1}{R_2 C} = 10^4$ *from which* $R_2 = \frac{10^{-4}}{C} = 10^3$. *So* $R_1 = 9\,\mathrm{k}$ *and* $R_2 = 1\,\mathrm{k}$.

Several people wrongly multiplied the corner frequencies by 2π. Quite a few used the wrong corner frequency when calculating the resistor values and so ended up with resistors that were 10 times their correct values. Quite a lot of people make $R_1 = 10\,\mathrm{k}$ instead of $9\,\mathrm{k}$.

---

g) The circuit of Figure 1.7 shows a 50 Hz voltage source with RMS voltage $\widetilde{V} = 230$ driving a load of impedance $Z_L = 15 + 10j\,\Omega$ through a line of impedance $Z_T = 0.3 + 1.5j\,\Omega$. Calculate the complex power absorbed by (i) Z_T and (ii) Z_L. [5]

The current phasor is therefore $\widetilde{I}_L = \frac{230}{Z_L + Z_T} = \frac{230}{15.3 + 11.5j} = 9.606 - 7.22j$. *From this* $\left|\widetilde{I}_L\right| = 12.02$ *and* $\left|\widetilde{I}_L\right|^2 = 144.4$.

The complex power absorbed by Z_L *is*

$$\left|\widetilde{I}_L\right|^2 Z_L = 144.4\,(15 + 10j) = 2166 + 1444j\,\mathrm{VA} = 2603\angle 33.7^\circ$$

.

The complex power absorbed by Z_T *is*

$$\left|\widetilde{I}_L\right|^2 Z_T = 144.4\,(0.3 + 1.5j) = 43.3 + 216.6j\,\mathrm{VA} = 221\angle 78.7^\circ$$

.

If you want to use the $\widetilde{V} \times \tilde{I}^*$ *formula instead, you must calculate*

$$\widetilde{V}_L = (9.606 - 7.22j)\,(15 + 10j) = 216.3 - 12.2j$$

and

$$\widetilde{V}_T = (9.606 - 7.22j)\,(0.3 + 1.5j) = 230 - \widetilde{V}_L = 13.7 + 12.2j$$

. *From this we get the absorbed powers as*

$$(216.3 - 12.2j)\,(9.606 + 7.22j) = 2166 + 1444j\,\mathrm{VA}$$

and

$$(13.7 + 12.2j)\,(9.606 + 7.22j) = 43.3 + 216.6j\,\mathrm{VA}.$$

Most people did this correctly. Many people wrote the correct formula $S = \left|\tilde{I}\right|^2 Z$ but actually used the formula $S = \tilde{I}^2 Z$ instead. A few used the correct formula $S = \frac{|\tilde{V}|^2}{Z^*}$ but either forgot to square the voltage, forgot to take magnitude or assumed it equaled $230\tilde{V}$. Several people used the incorrect (and dimensionally wrong) formula $S = \frac{|\tilde{I}|^2}{Z}$. Some said the currents through the two impedances were different: $\frac{230}{0.3+1.5j}$ and $\frac{230}{15+10j}$ respectively.

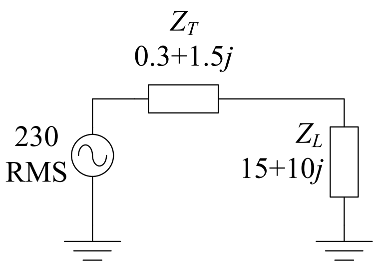


Figure 1.7

h) Figure 1.9 shows a transmission line of length 100 m that is terminated in a resistive load, R, with reflection coefficient $\rho = -0.6$. The line has a propagation velocity of $u = 2 \times 10^8$ m/s. At time $t = 0$, a forward-travelling (i.e. left-to-right) pulse arrives at X with amplitude 5 V and duration 1 μs as shown in Figure 1.8.

Draw a dimensioned sketch of the waveform at Y, a point 40 m from the end of the line, for $0 \le t \le 2\,\mu$s. Assume that no reflections occur at point X. [5]

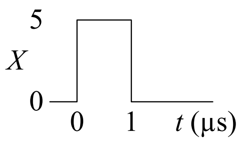


Figure 1.8

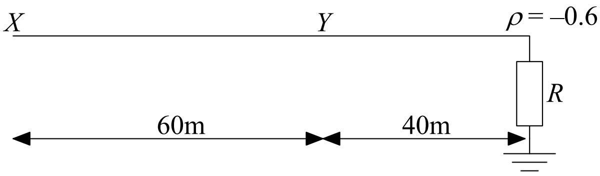
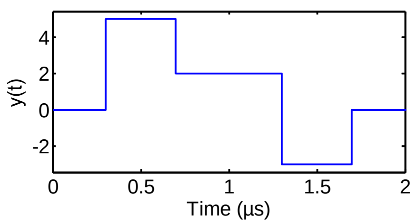


Figure 1.9

The forward wave takes $0.3\,\mu$s *to reach Y and a further* $0.4\,\mu$s *to reflect from the end and return to Y. Therefore the waveform at Y is the sum of two overlapping waves: (i) a pulse of amplitude* 5 V *beginning at* $t = 0.3\,\mu$s *and a pulse of* $5\rho = -3$ V *beginning at* $t = 0.7\,\mu$s*. Where the pulses overlap, their combined voltage is* $5 - 3 = 2$ V.



A few people got the adder correctly worked out the waveforms of the forward and backward waves at point Y but did not add them together to give the actual voltage waveform.

2.

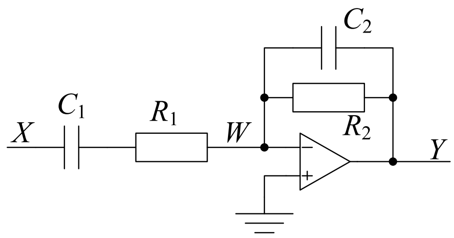


Figure 2.1

a) Assuming that the op-amp in the circuit of Figure 2.1 is ideal, show that its transfer function is given by [5]

$$\frac{Y}{X}(j\omega) = \frac{-j\omega R_2 C_1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1)}$$

.

This is an inverting amplifier whose gain is therefore $\frac{Y}{X} = \frac{-Z_2}{Z_1}$. Alternatively, using nodal analysis, KCL at W and assuming $W = 0$ we can derive $\frac{0-X}{Z_1} + \frac{0-Y}{Z_2} = 0 \quad \Rightarrow \quad \frac{Y}{X} = \frac{-Z_2}{Z_1}$.

For this circuit, $Z_1 = R_1 + \frac{1}{j\omega C_1} = \frac{j\omega R_1 C_1 + 1}{j\omega C_1}$ and $Z_2 = \frac{1}{\frac{1}{R_2} + j\omega C_2} = \frac{R_2}{j\omega R_2 C_2 + 1}$. Hence $\frac{Y}{X}(j\omega) = \frac{-Z_2}{Z_1} = \frac{-j\omega R_2 C_1}{(j\omega R_1 C_1 + 1)(j\omega R_2 C_2 + 1)}$.

Almost everyone got this right although sometimes requiring several pages of algebra. Surprisingly, a few "proved" the wrong expression, with $-j\omega R_2 C_2$ in the numerator. Others omitted the minus sign in the result. Some people multiplied out the denominator and then factorized it again to get the required answer which is a lot of unnecessary effort. Some introduced an additional unknown node voltage between C_1 and R_1 rather than treating these as a single complex impedance; this makes the nodal analysis more complicated.

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b) Assuming that $R_1 C_1 \gg R_2 C_2$, sketch straight-line approximations of (i) the magnitude response and (ii) the phase response of the circuit.

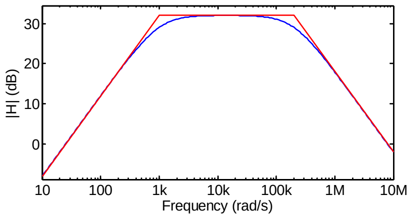
Indicate on the plots the frequency and value at each point where the gradient changes. [6]

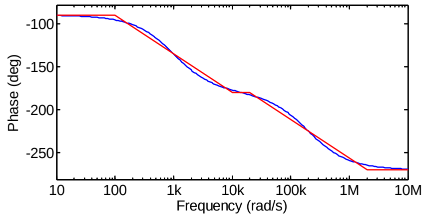
The LF and HF asymptotes are $-j\omega R_2 C_1$ and $\frac{-1}{j\omega R_1 C_2}$ respectively and the corner frequencies are $\frac{1}{R_1 C_1}$ and $\frac{1}{R_2 C_2}$ (they are in that order since you are told $R_1 C_1 \gg R_2 C_2$). Between the two corner frequencies, the gain is $\frac{-R_2}{R_1}$; this can be obtained by substituting $\omega = \frac{1}{R_1 C_1}$ into the LF asymptote (or $\omega = \frac{1}{R_2 C_2}$ into the HF asymptote). The phases of the asymptotes are $-90°$ and $+90°$ (or $-270°$) respectively with a mid-frequency phase of $\pm 180°$.

Several people thought that since $R_1 C_1 \gg R_2 C_2$ you could completely ignore the $(j\omega R_2 C_2 + 1)$ factor leading to a high frequency asymptote of $-\frac{R_2}{R_1}$ (or oc-

casionally ignore the other factor giving an asymptote of $-\frac{C_1}{C_2}$). The grounds for this seemed to be an incorrect belief that $R_1C_1 \gg R_2C_2 \Rightarrow \omega R_2C_2 \ll 1$. Several thought the low frequency asymptote phase was $+90^\circ$ due to the factor j but ignored the minus sign which changes the phase by $\pm 180^\circ$. Some gave the corner frequencies as the dimensionally incorrect R_1C_1 and R_2C_2.

With the values from part (c), the magnitude and phase plots therefore look like this:





For the magnitude plot the corners are at $\left(\frac{1}{R_1C_1}, \frac{R_2}{R_1}\right)$ *and* $\left(\frac{1}{R_2C_2}, \frac{R_2}{R_1}\right)$.

Several people got the corner frequencies in the wrong order: $R_1C_1 \gg R_2C_2 \Leftrightarrow \frac{1}{R_1C_1} \ll \frac{1}{R_2C_2}$. Some introduced a third corner frequency at $\omega = \frac{1}{R_2C_1}$ presumably because of the numerator $-j\omega R_2C_1$; you only get corner frequencies from factors like $(aj\omega+b)$ or $\left(a(j\omega)^2+bj\omega+c\right)$. Some said the gain in the flat portion was proportional to ω which is self-contradictory; others had a variety of dimensionally incorrect values (e.g. R_1C_1 or R_2C_1); a voltage gain must always be dimensionless.

For the phase plot, the corners are at $\left(\frac{0.1}{R_1C_1}, -90^\circ = -\frac{\pi}{2}\right)$, $\left(\frac{10}{R_1C_1}, -180^\circ = -\pi\right)$, $\left(\frac{0.1}{R_2C_2}, -180^\circ = -\pi\right)$ *and* $\left(\frac{10}{R_2C_2}, -270^\circ = -\frac{3\pi}{2}\right)$. *The condition* $R_1C_1 \gg R_2C_2$ *ensures the corner frequencies are in the order given above and that there is therefore a flat portion in the middle of the phase plot for* $\frac{10}{R_1C_1} < \omega < \frac{0.1}{R_2C_2}$. *Each sloping portion of the graph goes down by* 90° *(or* $\frac{\pi}{2}$ *radians) over the space of two decades in frequency (= a factor of* 100*); thus the gradient is* 45° *per decade. The low and high frequency asymptotes in the phase plot always have zero gradient.*

Some added a sudden jump of 2π at the end of the phase plot to make the final phase equal to $+\frac{\pi}{2}$. This is not incorrect, but it is unnecessary: phase shifts of $+\frac{\pi}{2}$ and $-\frac{3\pi}{2}$ mean the same thing. Others, possibly for the same reason, had the entire phase graph inverted. Many people put no labels on the vertical axes of one or both graphs. Quite a few made the phase plot bend up at $\frac{0.1}{R_2C_2}$ instead of down; if the coefficients in a linear factor have the same sign as each other, the first bend is up for the numerator and down for the denominator.

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c) If $R_2 = 40\,\text{k}\Omega$, determine values for R_1, C_1 and C_2 to give corner frequencies at $1\,\text{krad/s}$ and $200\,\text{krad/s}$ and a magnitude gain of 40 in the horizontal portion of the magnitude response. [5]

The mid-frequency magnitude gain is $\frac{R_2}{R_1} = 40 \quad \Rightarrow \quad R_1 = \frac{R_2}{40} = 1\,\text{k}\Omega$.

Some took the question to mean a magnitude gain of $40\,\text{dB}$ instead of 40. Many found it difficult to determine the formula for the mid-frequency gain; you can get it by substituting the first corner frequency, $\frac{1}{R_1C_1}$, into the LF asymptote *or the second corner frequency, $\frac{1}{R_2C_2}$, into the HF asymptote.*

The LF corner is $\frac{1}{R_1C_1} = 1000 \quad \Rightarrow \quad C_1 = \frac{1}{1000R_1} = 1\,\mu\text{F}.$

The HF corner is $\frac{1}{R_2C_2} = 200000 \quad \Rightarrow \quad C_2 = \frac{10^{-6}}{0.2R_2} = 125\,\text{pF}.$

Many people got the corner frequencies in the wrong order even if they had shown them correctly on the graph earlier: $R_1C_1 \gg R_2C_2 \Leftrightarrow \frac{1}{R_1C_1} \ll \frac{1}{R_2C_2}$.

Calculate the actual gain in dB at each of the two corner frequencies and compare these with the values predicted by the straight-line approximation. [3]

From part (a) we know $\frac{Y}{X}(j\omega) = \frac{-j\omega R_2C_1}{(j\omega R_1C_1+1)(j\omega R_2C_2+1)} = \frac{-\left(4\times10^{-2}\right)j\omega}{(10^{-3}j\omega+1)((5\times10^{-6})j\omega+1)}$.

At $\omega = 1000$, $\frac{Y}{X}(j\omega) = \frac{-40j}{(1+j)(1+0.005j)} = -20.1 - 19.9j = 28.3\angle -135.3^\circ = 29\,\text{dB}.$

At $\omega = 2\times10^5$, $\frac{Y}{X}(j\omega) = \frac{-8000j}{(1+200j)(1+j)} = -20.1 + 19.9j = 28.3\angle 135.3^\circ = 29\,\text{dB}.$

The straight-line approximation predicts a gain of $40 = 32\,\text{dB}$ *so the error is* $-3\,\text{dB}$ *in each case.*

Many people omitted this part completely. Very many people just ignored j when evaluating arithmetic expressions. You cannot do this even if you only want the absolute value of the expression because, for example, $|1+j| = 1.414$ *but* $1+|j| = 2$. *Others misread the numerator as* $-j\omega R_2C_2$ *and got answers that were too low by a factor of* 200.

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d) Determine the transfer function $\frac{Y}{X}$ of the circuit of Fig. 2.2 under the assumption that the op-amp has a finite gain, $A = -\frac{Y}{W}$, but is otherwise ideal. [3]

KCL at node W gives $\frac{W-X}{Z_1} + \frac{W-Y}{Z_2} = 0 \quad \Rightarrow \quad W(Z_1+Z_2) - XZ_2 - YZ_1 = 0.$ *If the opamp has gain A, then* $Y = -AW$ *which gives* $W = \frac{-Y}{A}$. *Substituting this into the previous expression gives* $-Y(Z_1+Z_2) - XAZ_2 - YAZ_1 = 0$ *from which* $Y(Z_1(1+A)+Z_2) = -AZ_2X$ *and so* $\frac{Y}{X} = \frac{-AZ_2}{Z_1(1+A)+Z_2}$.

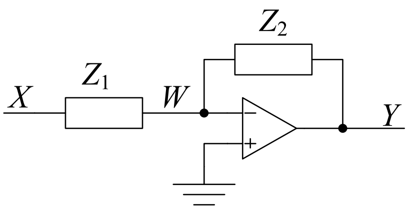

Figure 2.2

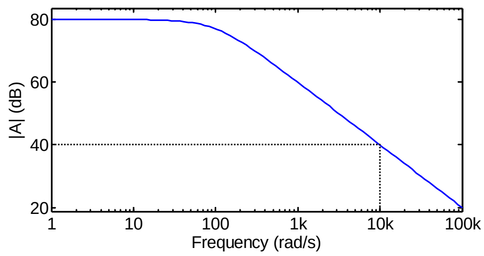

Figure 2.3

With a finite gain A we have $Y = -AW$. *We can apply KCL at node W and take*

$W = \frac{-Y}{A}$ to obtain $\frac{-Y-AX}{Z_1} + \frac{-Y-AY}{Z_2} = 0 \quad \Rightarrow \quad Y(Z_1(1+A)+Z_2) = -XAZ_2$ from which $\frac{Y}{X} = \frac{-AZ_2}{Z_1(1+A)+Z_2}$.

Note that assuming $V_+ = V_-$ is the same as assuming the op-amp gain is infinite. Many people assumed this and obtained the solution for an ideal op-amp: $\frac{Y}{X} = -\frac{Z_2}{Z_1}$ thereby missing the entire point of the question. Quite a few people made an algebraic error in the final stage and ended up with the gain expression inverted.

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e) The op-amp magnitude response is $|A(j\omega)| = \left|\frac{A_0\omega_0}{j\omega+\omega_0}\right|$ and is plotted in decibels in Fig. 2.3. Estimate the values of A_0 and ω_0 from the plot. [3]

The low frequency asymptote is $A_0 = 80\,\text{dB} = 10^4$.

Many people took the gain to be 80 rather than 80 dB; 80 is a very low DC gain for an op-amp. Alternatively, they tried to do the algebra while keeping A_0 in dB deliberately; this doesn't work - you must convert decibel values into actual gains when doing algebra on transfer functions. A few took the LF asymptote to be $A_0\omega_0$.

The high frequency asymptote is $\frac{A_0\omega_0}{j\omega} = \frac{10^4\omega_0}{j\omega}$. From the graph we see that $A(j10^4) = 40\,\text{dB} = 100$. Substituting this into the asymptote expression gives $\frac{10^4\omega_0}{j10^4} = 100$ from which $\omega_0 = 100$.

Some found this very easy and just wrote down the answers; others got involved in loads of horrible simultaneous equations and usually ended up with the wrong answer. The advantage of using straight line approximations (including the low and high frequency asymptotes) is that you avoid all the horrible algebra. Some did not notice that the frequency axis was in rad/s and gave $\omega_0 = 2\pi\times 100$. Some wrote equations in which the gains were in dB, e.g. $A = 40 = \left|\frac{80\omega_0}{j\omega+\omega_0}\right|$ when $\omega = 10000$; you must convert dB values to true gains before using them in algebraic expressions. As elsewhere, people frequently ignored the j in complex-valued expressions. A few ended up with complex values for A_0 and/or ω_0 because they omitted the modulus signs in an expression.

---

f) Determine the gain in dB of the circuit of Fig. 2.1 at (i) $\omega = 1$ krad/s and (ii) $\omega = 200$ krad/s if the opamp open loop gain is $A(j\omega) = \frac{A_0\omega_0}{j\omega+\omega_0}$ with the values of A_0 and ω_0 as determined in part (e). [5]

From the answer to part (d)) we know $\frac{Y}{X} = \frac{-AZ_2}{Z_1(1+A)+Z_2}$ and from (e) $A(j\omega) = \frac{10^6}{j\omega+100}$.

We also have $Z_1 = \frac{j\omega R_1C_1+1}{j\omega C_1} = \frac{1+10^{-3}j\omega}{10^{-6}j\omega} = 1000 + \frac{1000}{j\omega}$ and $Z_2 = \frac{R_2}{j\omega R_2C_2+1} = \frac{4\times 10^4}{1+5j\omega\times 10^{-6}}$.

(i) At $\omega = 1000$, $A = \frac{10^6}{100+1000j} = 99-990j = 995\angle-84^\circ = 60\,\text{dB}$, $Z_1 = 1000 + \frac{1000}{1000j} = 1000-1000j$, $Z_2 = \frac{4\times10^4}{1+5j\times10^{-3}} = 4\times10^4 - 200j$.

Substituting these values in give

$$\frac{-(99-990j)\left(4\times10^4-200j\right)}{(1000-1000j)(100-990j)+4\times10^4-200j} = \frac{(-3.76+39.6)\times10^6}{(-0.85-1.09)\times10^6}$$
$$= -20.93-19.77j$$
$$= 28.79\angle-136^\circ = 29.2\,\text{dB}$$

which is slightly more than the ideal op-amp value.

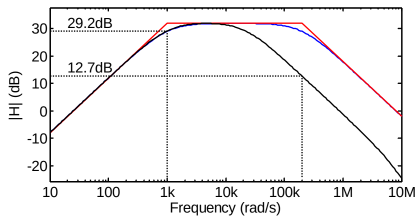
(ii) At $\omega = 2\times10^5$, $A = \frac{10^6}{100+2j\times10^5} = 0.0025-5j = 5\angle-89.97^\circ = 14\,\text{dB}$, $Z_1 = 1000+\frac{1000}{2j\times10^5} = 1000-5j$, $Z_2 = \frac{4\times10^4}{1+j} = (2-2j)\times10^4$.

Substituting these values in give

$$\frac{-(0.0025-5j)(2-2j)\times10^4}{(1000-5j)(1.0025-5j)+(2-2j)\times10^4} = \frac{(0.999+1j)\times10^5}{(2.1-2.5)\times10^4}$$
$$= -0.38+4.32j$$
$$= 4.33\angle95^\circ = 12.7\,\text{dB}$$

which is much less than the ideal op-amp value.

Although not requested, the finite-A curve is plotted below along with the ideal curve and the straight line approximation.



Several people just worked out the gain of the op-amp rather than the gain of the circuit in Fig. 2.1. Some used the value of A in dB within the equations instead of its numerical value of 10 000. If you are not systematic, this part involves a lot of complex arithmetic and many people did not attempt it.

3. In the circuit shown in Fig. 3.1, the switch is turned rapidly on and off in order to control the average inductor current, I_L. In the switch waveform illustrated in Fig. 3.2, the switch is repeatedly closed for $T_C = 0.5$ and then opened for $T_O = 1$ where, in this question, all times are in milliseconds.

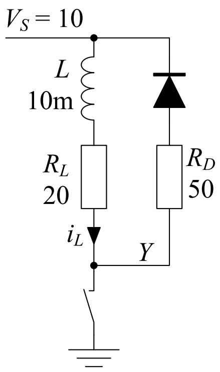


Figure 3.1

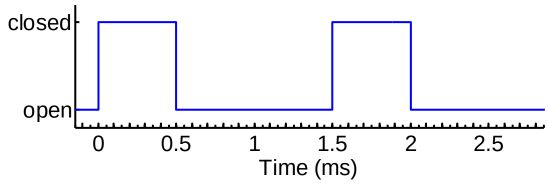


Figure 3.2

a) For intervals when the switch is closed, determine the time constant, τ_C, in milliseconds and the steady-state inductor current, I_C. [3]

When the switch is closed, node Y is grounded and the diode and R_D do not affect i_L at all. The time constant is $\tau_C = \frac{L}{R_L} = 0.5\,\text{ms}$.

The steady state current can be obtained by taking the inductor to be a short circuit. This gives a current of $I_C = 500\,\text{mA}$.

A few people took $V_S = 10$ to be a phasor and then wrote the phasor equation $I_C = \frac{10}{R_L + j\omega L}$ usually going on to assume that $\omega = 1$ or else $\omega = \frac{2000\pi}{1.5}$. You cannot possibly use phasors in this problem because every time the switch operates, the circuit itself changes. Note too that this circuit does not have an input signal and so does not have a "gain".

---

b) If the switch is closed at time $t = 0$ and opened again at $t = 0.5$ as shown in Fig. 3.2, determine an expression for $i_L(t)$ for $0 < t < 0.5$ given that $i_L(0) = 0$. [5]

The current is given by $i_L(t) = I_C + Ae^{-\frac{t}{\tau_C}}$ where A is determined by the initial condition $i_L(0) = 0$. Substituting this in gives $0 = I_C + A \quad \Rightarrow \quad A = -I_C$.

Putting this together gives $i_L(t) = 0.5\left(1 - e^{-2t}\right)$ where t is in ms.

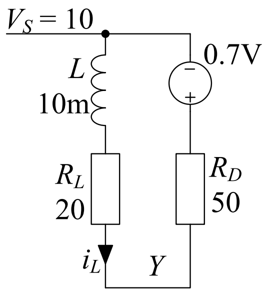
A few people took $i_L(0) = \frac{10}{70} = 0.143\,\text{A}$ as the [incorrect] steady state when the switch was open even thought he question explicitly says $i_L(0) = 0$.

---

c) Show that when the switch is open and the diode is forward biased, the steady state inductor current, I_O, is equal to $-10\,\text{mA}$. Assume that the forward voltage drop of the diode is $0.7\,\text{V}$. [2]

Determine the time constant, τ_O, when the switch is open and the diode is forward biased. [3]

If the diode is forward biased, it acts as a voltage source and so, when the switch is open, the circuit becomes



The time constant is therefore $\tau_O = \frac{L}{R_L + R_D} = 0.143\,\text{ms}$ *and the steady state current is* $I_O = \frac{-0.7}{R_L+R_D} = \frac{-0.7}{70} = -10\,\text{mA}$.

Note that the steady state current flows in the direction opposite to the arrow; several people got the diode voltage the wrong way around. Several people "proved" that the steady state current was $+10\,\text{mA}$ *by being rather vague about which direction they were measuring current. A very small number of people calculated a super-accurate time constant as* $\frac{L}{R_L+R_D+r_d}$ *where the small signal diode resistance was* $r_d = \frac{V_T}{i_L}$. *The only difficulty with this is that the value of* i_L *and hence of* r_d *will not vary with time. Luckily* r_d *makes rather little difference: at the peak current of* $0.316\,\text{A}$, $r_d = 0.79\,\Omega$.

d) If the switch waveform is as shown in Fig. 3.2, determine the time, T_Z, when $i_L(t)$ first reaches zero after the switch opens at $t = 0.5$ and determine an expression for $i_L(t)$ over the interval $0.5 < t < T_Z$. [6]

From the answer to part (b), $i_L(0.5) = 0.5\left(1 - e^{-1}\right) = 0.316$.

Some mixed up different time units using seconds for the time constant but milliseconds for the time e.g. $e^{-\frac{0.5}{0.0005}}$; *this is asking for trouble.*

When the switch is open, the current is given by $i_L(t) = I_O + Be^{-\frac{t-0.5}{\tau_O}}$ *where* B *is determined by the initial condition* $i_L(0.5) = 0.316$. *Substituting this in gives* $0.316 = I_O + B = -0.01 + B \quad \Rightarrow \quad B = 0.326$.

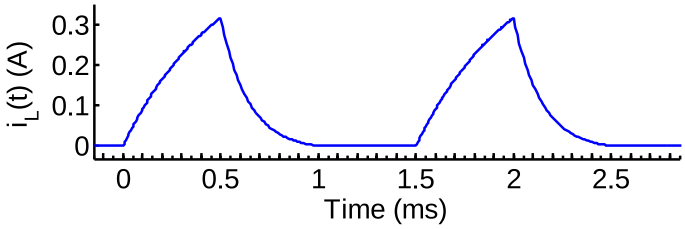
Several people added $10\,\text{mA}$ *to* $0.316\,\text{A}$ *by calculating* $10 + 0.316 = 10.316$ *in somewhat vague output units; I strongly recommend that you convert all voltages, currents and impedances into the same units before doing arithmetic on them. Some got the sign wrong and ended up with a transient amplitude of* $0.316 - 10 = 0.306\,\text{mA}$. *If you substitute* $t = 0.5$ *into the final expression (below), you should obtain the expected value of* $i_L(0.5) = 0.316$; *this is a good check. Several people calculated* I_0 *correctly in part (c) but then inexplicably used the value* $I_0 = 0$ *in this part.*

Putting this together gives $i_L(t) = 0.326e^{-7(t-0.5)} - 0.01$ *where* t *is in ms. You can also write this as* $i_L(t) = 10.8e^{-7t} - 0.01$ *where* $10.8 = 0.326e^{-7\times -0.5}$ *although it is less illuminating.*

This equals zero when

$t = 0.5 + \frac{1}{7}\ln\left(\frac{0.326}{0.01}\right) = 0.5 + \frac{3.484}{7} = 0.5 + 0.498 = 0.998\,\text{ms}.$

Although not requested, a plot of $i_L(t)$ is shown below:



Many people thought the current would go negative and almost reach the steady state of $-10\,mA$. In fact, as soon as the current reaches zero, the diode turns off and the current stays at zero.

---

e) Draw a dimensioned sketch showing the voltage waveform, $y(t)$, at node Y over the interval $0 < t < 2.1$. Determine the peak value of $y(t)$. [6]

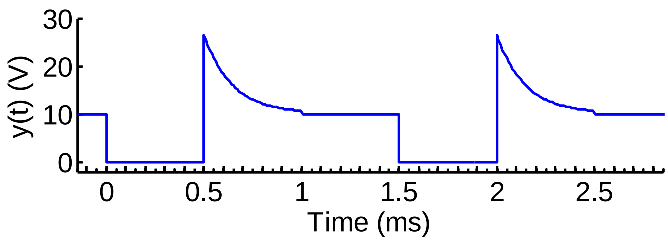
There are three separate situations to consider:

(i) when the switch is closed, $y(t) = 0$,

(ii) when the switch is open and the diode is forward biased, $y(t) = 10.7 + 50i_L(t)$ from the equivalent circuit above and

(iii) when the switch is open and the diode is off, then $i_L = 0$ and $y(t) = V_S = 10$.

At $t = 0.5$ therefore, $i_L = 0.316$ and so the peak voltage is $10.7 + 50 \times 0.316 = 26.5\,\text{V}$. The complete waveform of $y(t)$ looks like this and includes an abrupt jump from 10.7 to $10\,\text{V}$ when the diode turns off at $t = 0.998$.



Quite a few people misread the question and drew a graph of $i_L(t)$ instead. When the switch is open, you can calculate the voltage at Y relative to the $10\,\text{V}$ supply in two ways (i) the sum of the voltages across the diode and the $50\,\Omega$ resistor or (ii) the sum of the voltages across the inductor and the $20\,\Omega$ resistor. The second way is much more effort because it depends on the derivative of the current; some people tried to do it this way nevertheless.

---

f) Suppose now that $T_O = 0.2$ as shown in Fig. 3.3. The current, $i_L(t)$ will now oscillate between the values I_A at $t = 0, 0.7, \cdots$ and I_B at $t = 0.5, 1.2, \cdots$.

Determine the values of I_A and I_B. [5]

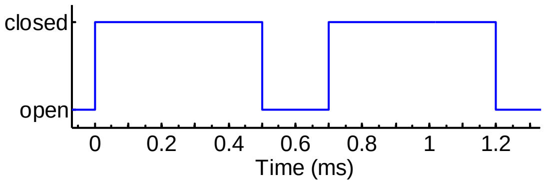


Figure 3.3

When the switch is closed, we now have $i_L(t) = I_C + (I_A - I_C)e^{-2t}$ *and so* $i_L(0.5) = I_B = I_C + (I_A - I_C)e^{-1} = 0.316 + 0.368 I_A$.

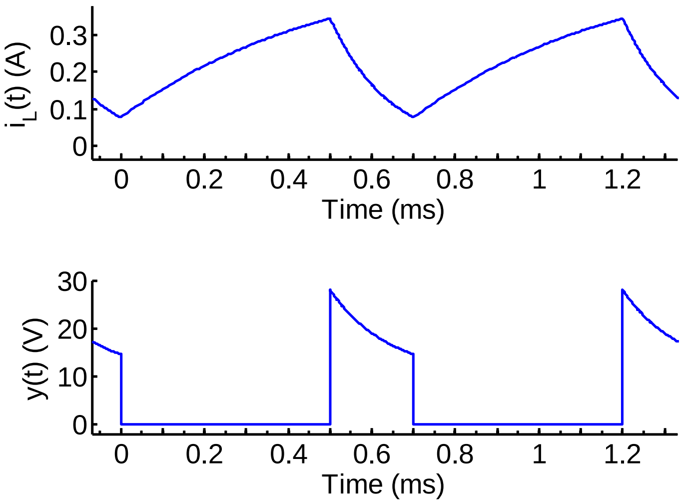
When the switch is opened, we have $i_L(t) = I_O + (I_B - I_O)e^{-7(t-0.5)}$ *and so* $i_L(0.7) = I_A = I_O + (I_B - I_O)e^{-1.4} = -0.0075 + 0.2466 I_B$.

Substituting the first equation into the second gives

$I_A = -0.0075 + 0.07793 + 0.0907 I_A \quad \Rightarrow \quad I_A = \frac{0.0704}{0.9093} = 0.0774\,\text{A}$

Hence $I_B = 0.316 + 0.368 I_A = 0.316 + 0.0285 = 0.3445\,\text{A}$.

Although not requested, the waveforms are



Only a few people understood the concept of this question. Most assumed that $i_L(0) = 0$ *even thought the question stated otherwise.*
