IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2015

EEE/EIE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Friday, 5 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible	First Marker(s) :	D.M. Brookes
	Second Marker(s) :	P. Georgiou

ANALYSIS OF CIRCUITS

Information for Candidates:

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

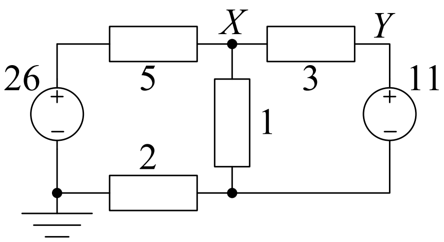

Figure 1.1

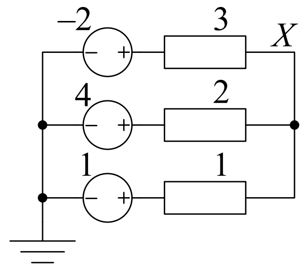

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the values of its components. [5]

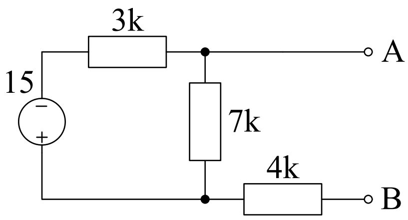

Figure 1.3

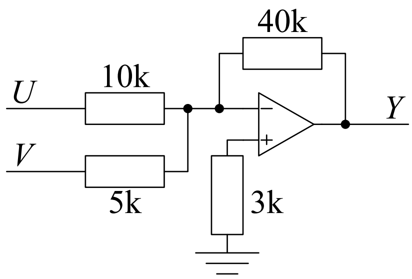

Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

e) The waveform, $x(t)$, is a periodic triangle wave of amplitude $\pm 4\,\text{V}$ as shown in Figure 1.5. The waveform is applied to the input, X, of the circuit shown in Figure 1.6. The diode has a forward voltage drop of $0.7\,\text{V}$ and is otherwise ideal.

Determine the maximum and minimum values of the waveform $y(t)$ and determine the input voltage, x_0, at which the diode turns on. [5]

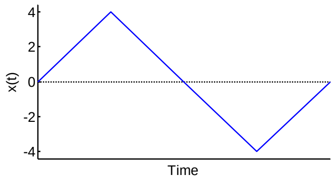

Figure 1.5

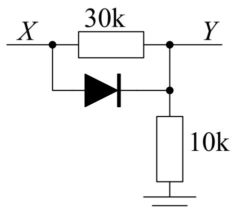

Figure 1.6

f) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are V and W respectively. [5]

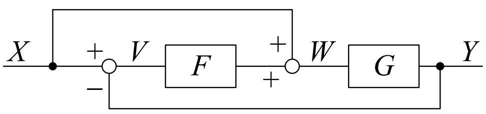

Figure 1.7

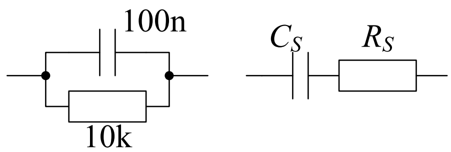

Figure 1.8

g) i) Determine C_S and R_S so that the two networks in Figure 1.8 have the same impedance at $\omega_0 = 2000\,\text{rad/s}$.

ii) Using logarithmic axes for both frequency and impedance sketch a graph showing the impedance magnitude of both networks for the frequency range $20 < \omega < 200000$. [5]

h) The waveform, $x(t)$, shown in Figure 1.9 is applied to the input, X, of the circuit shown in Figure 1.10. Determine the time constant of the circuit and the amplitude of the transient component of $y(t)$.

Hence draw a dimensioned sketch of the waveform at Y . [5]

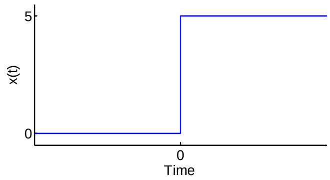

Figure 1.9

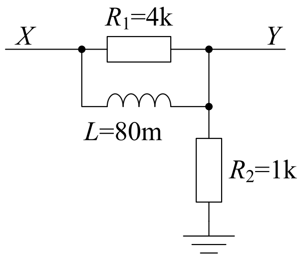

Figure 1.10

2. A second order transfer function is given by

$$H(j\omega) = \frac{-G}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$$

where G, ω_0 and ζ are positive real numbers.

a) Determine the magnitude and phase of $H(j\omega)$ at [4]

i) $\omega = 0$,

ii) $\omega = \omega_0$,

iii) $\omega \gg \omega_0$.

b) If we define $\phi(\omega) = \angle H(j\omega)$, show that $\phi(\omega) = \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega^2-\omega_0^2}\right)$ and hence show that its derivative at ω_0 equals $\phi'(\omega_0) = \frac{-1}{\zeta\omega_0}$. [6]

c) Suppose that $G = 5$, $\zeta = 0.8$ and $\omega_0 = 10^4\,\mathrm{rad/s}$.

i) Sketch a dimensioned graph of $|H(j\omega)|$ in decibels using a logarithmic frequency axis. Your graph should include a sketch of the true magnitude response in addition to the high and low frequency asymptotes. [3]

ii) Sketch a dimensioned graph of $\angle H(j\omega)$ using a linear phase axis in radians and a logarithmic frequency axis. [3]

d) Fig. 2.1 shows the circuit diagram of a filter circuit. Assuming the opamp to be ideal, use nodal analysis to show that the frequency response of the filter is given by

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-R_2}{R_1R_2R_3C_1C_2(j\omega)^2 + (R_1R_2 + R_1R_3 + R_2R_3)C_1 j\omega + R_1}.$$

[6]

e) Find expressions for G, ω_0 and ζ in terms of the component values when the frequency response of the filter is expressed in the form given for $H(j\omega)$ above. [4]

f) If $R_2 = 60\,\mathrm{k\Omega}$ and $R_3 = \frac{50}{3}\,\mathrm{k\Omega}$ determine values for R_1, C_1 and C_2 such that $G = 5$, $\zeta = 0.8$ and $\omega_0 = 10^4\,\mathrm{rad/s}$. [4]

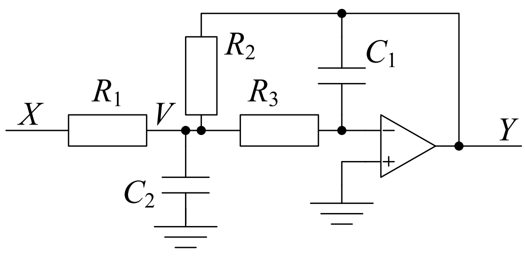

Figure 2.1

3. The circuit of Fig. 3.1 shows a transmission line of length L driven by a sinusoidal voltage source $v_S(t)$ through a resistor, R_S. The characteristic impedance and propagation velocity of the line are Z_0 and u respectively. The phasor corresponding to the waveform $v_S(t)$ is written V_S and similarly for other waveforms.

The voltage and current waveforms at a distance x from the source are given respectively by

$$\begin{aligned} v_x(t) &= f_x(t) + g_x(t) \\ i_x(t) &= Z_0^{-1}\left(f_x(t) - g_x(t)\right) \end{aligned}$$

where $f_x(t) = f_0\left(t - u^{-1}x\right)$ and $g_x(t) = g_0\left(t + u^{-1}x\right)$ are the forward and backward waves at a distance x from the source.

a) Show that if $f_0(t) = A\cos(\omega t + \phi)$ then the phasors F_x and F_0 satisfy

$$F_x = F_0 e^{-j\omega u^{-1}x}.$$

Determine a similar expression relating G_x and G_0. [5]

You may assume without proof that the phasor corresponding to $A\cos(\omega t + \psi)$ is $Ae^{j\psi}$.

b) Use the load equation $V_L = I_L R_L$ to show that G_0 can be written in the form $G_0 = \rho_L e^{j\theta} F_0$ and determine expressions for the real-valued constants ρ_L and θ. [5]

c) By applying Kirchoff's current law at the point marked $v_0(t)$ in Fig. 3.1, show that F_0 may be expressed as $F_0 = \tau_S V_S + \rho_S G_0$ and determine expressions for the real-valued constants τ_S and ρ_S. [5]

d) Eliminate G_0 between the answers to parts b) and c) to obtain an expression for F_0 in terms of V_S. [4]

e) Suppose that $R_S = 25\,\Omega$, $R_L = 400\,\Omega$, $Z_0 = 100\,\Omega$, $L = 10\,\text{m}$, $u = 1.5\times10^8\,\text{m/s}$, $V_S = 10j$ and $\omega = 6\times10^7\,\text{rad/s}$.

Determine the phasors V_0 and I_0. [6]

f) Calculate the complex power supplied by V_S and the average power absorbed by R_S. Hence deduce the average power absorbed by R_L. [5]

Figure 3.1

ANALYSIS OF CIRCUITS

****** Solutions 2015 ******

Information for Candidates:

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Key: B=bookwork, U=unseen example

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [5]

[U] We can label the voltages on the top left and bottom right nodes as 26 and $Y-11$ respectively. We now write down KCL equation at node X to obtain

$$\begin{aligned} \frac{X-26}{5}+\frac{X-Y+11}{1}+\frac{X-Y}{3} &= 0 \\ \Rightarrow 3X-78+15X-15Y+165+5X-5Y &= 0 \\ \Rightarrow 23X-20Y &= -87 \end{aligned}$$

KCL at the Y supernode gives

$$\begin{aligned} \frac{Y-11}{2}+\frac{Y-11-X}{1}+\frac{Y-X}{3} &= 0 \\ \Rightarrow 3Y-33+6Y-66-6X+2Y-2X &= 0 \\ \Rightarrow -8X+11Y &= 99 \end{aligned}$$

Combining these gives $253X-160X=-957+1980 \quad \Rightarrow \quad X=\frac{1023}{93}=11$

from which $11Y=99+88=187 \quad \Rightarrow \quad Y=\frac{187}{11}=17$.

Mostly done correctly except for the occasional algebra error (note that the calculators supplied in exams can solve simultaneous equations). The most common mistake was omitting the current through the $1\,\Omega$ resistor from one or both equations, i.e. the terms $\frac{X-(Y-11)}{1}$ and $\frac{(Y-11)-X}{1}$; it is essential to include every current path out of a node or super-node.

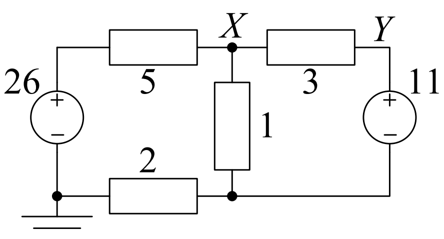

Figure 1.1

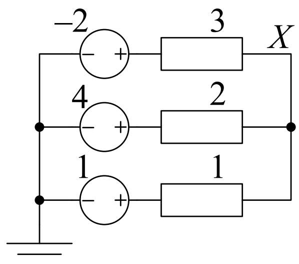

Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [5]

[U] If we short circuit the $-2\,\text{V}$ and $4\,\text{V}$ voltage sources, the $3\,\Omega$ and $2\,\Omega$ resistors are in parallel and equal $\frac{6}{5}\,\Omega$. We therefore have a potential divider and $X=1\times\frac{\frac{6}{5}}{1+\frac{6}{5}}=\frac{6}{11}\,\text{V}$.

If we short circuit the $-2\,\text{V}$ and $1\,\text{V}$ voltage sources, the $3\,\Omega$ and $1\,\Omega$ resistors are in parallel and equal $\frac{3}{4}\,\Omega$. We therefore have a potential divider and $X=4\times\frac{\frac{3}{4}}{2+\frac{3}{4}}=\frac{12}{11}\,\text{V}$.

If we short circuit the $4\,\mathrm{V}$ *and* $1\,\mathrm{V}$ *voltage sources, the* $2\,\Omega$ *and* $1\,\Omega$ *resistors are in parallel and equal* $\frac{2}{3}\,\Omega$*. We therefore have a potential divider and* $X = -2 \times \frac{\frac{2}{3}}{3+\frac{2}{3}} = \frac{-4}{11}\,\mathrm{V}$.

By superposition, the total voltage is therefore $\frac{6+12-4}{11} = \frac{14}{11} = 1.27\,\mathrm{V}$.

In most questions you are free to use any valid method to obtain the answer; however this question specifically requires you to use the method of superposition so if you solve the problem using nodal analysis you will get zero marks ☹. The whole point of superposition is that you find the contribution of each source in turn by setting all the remaining sources to zero. Quite a few people set only one of sources to zero each time (leaving two at their original values). A few people open-circuited the unwanted voltage sources rather than short-circuiting them; the idea is to set their value to zero and a zero-valued voltage source is a short circuit.

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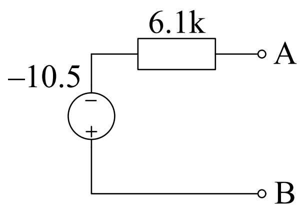
c) Draw the Thévenin equivalent circuit of the network in Figure 1.3 and find the values of its components. [5]

[U] We can find the open circuit voltage by ignoring the $4\,\mathrm{k}$ *resistor (since there is no current flowing through it). The circuit is a potential divider and the voltage across the* $7\,\mathrm{k}$ *resistor is* $V_{AB} = -15 \times \frac{7}{10} = -10.5\,\mathrm{V}$.

We can find the Thévenin resistance by short-circuiting the voltage source. The remaining network has a resistance of $4 + \frac{21}{10} = 6.1\,\mathrm{k}\Omega$.

So the complete Thévenin equivalent is:



Some people calculated the correct component values but did not actually draw the circuit as the questions demanded. When calculating the open-circuit voltage, several pwople took the voltage across the $3\,\mathrm{k}$ *resistor rather than the voltage across the* $7\,\mathrm{k}$ *resistor (which is what we need since it equals the voltage between B and A. Several people used nodal analysis to calculate the open-circuit voltage. If you use this method, you should define node B as ground (since you want to determine* V_{AB}*); the method is then made much easier if you notice that since there is no current through the* $4\,\mathrm{k}$ *resistor, there is no voltage drop across it and therefore the* $+$ *side of the voltage source is also at ground.*

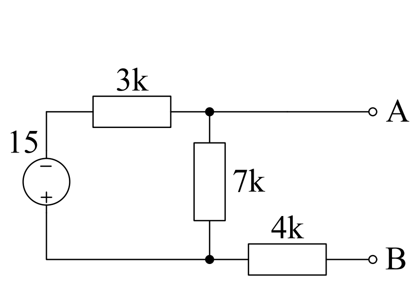


Figure 1.3

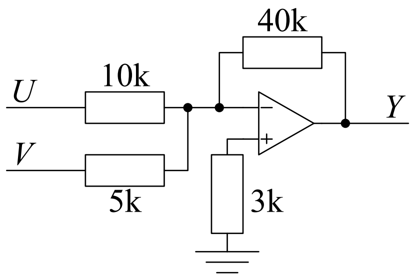


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Y in terms of U and V. [5]

[U] This is an inverting op-amp circuit and so we can write down $Y = \frac{-40}{10}U + \frac{-40}{5}V = -4U - 8V$.

The 3k *resistor has no effect because, since the op-amp is ideal, there is no current flowing through it. It is not however gratuitous; since* 3k *is approximately the parallel combination of the other three resistors, a non-zero bias current will have almost no effect on the output.*

Most people recognised this as an inverting op-amp circuit. A few used nodal analysis to calculate Y *which woks out easily provided you assume that negative feedback will force the* $-$ *input of the opamp to* 0V. *A few people forgot that it was inverting and made the gain positive.*

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e) The waveform, $x(t)$, is a periodic triangle wave of amplitude ± 4 V as shown in Figure 1.5. The waveform is applied to the input, X, of the circuit shown in Figure 1.6. The diode has a forward voltage drop of 0.7 V and is otherwise ideal.

Determine the maximum and minimum values of the waveform $y(t)$ and determine the input voltage, x_0, at which the diode turns on. [5]

[U] If the diode is off, then the circuit is a potential divider with $y = 0.25x$. *If the diode is on, then* $y = x - 0.7$.

The diode turns on at the input voltage when both of these conditions are true giving $0.25x = x - 0.7$ *from which* $0.75x = 0.7$*and hence* $x = 0.933$ V.

So the diode is on for $x > 0.933$ *and the maximum value of* y *will be* $y = 3.3$ *when* $x = +4$.

The diode is off for $x < 0.933$ *and the minimum value of* y *will be* -1 *when* $x = -4$.

Quite a lot of people thought the diode would turn on when $x = 0.7$ V; *it actually turns on when the voltage across the* 30 kΩ *resistor (i.e.* $0.75x$*) equals* 0.7 V *or, equivalently, when the conditions for the diode being on and being off are both true. Several assumed that, when the diode was conducting,* $Y = X$ *even though the question said that it had a voltage drop of* 0.7 V. *A few people negated the* *diode voltage; current flows in the direction of the arrow and must flow from $+$ to $-$ in any component that absorbs energy (just like a resistor). When the diode is forward biased (i.e. "on") it acts as a voltage source of $0.7\,\mathrm{V}$; you cannot then apply KCL at node Y because you do not know the current that is flowing through the diode (you don't of course need to do KCL either, since you know that $y = x - 0.7$). Several people applied KCL at node Y but omitted the diode current.*

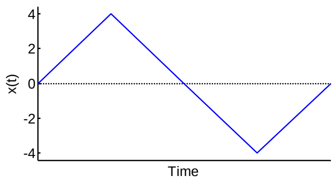


Figure 1.5

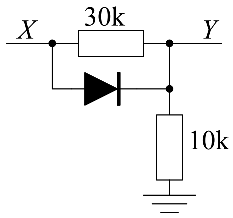


Figure 1.6

f) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.7. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F and G respectively. The open circles represent adder/subtractors; their inputs have the signs indicated on the diagram and their outputs are V and W respectively. [5]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= X - Y \\ W &= X + FV \\ Y &= GW \end{aligned}$$

We need to eliminate V and W from these equations:

$$\begin{aligned} Y &= GW \\ &= GX + FGV \\ &= GX + FG(X - Y) \\ &= GX + FGX - FGY \\ (1+FG)Y &= G(1+F)X \\ \frac{Y}{X} &= \frac{G+FG}{1+FG} \end{aligned}$$

In a linear block diagram such as this (i.e. no blocks that multiply two signals together) you never get two signals multiplied together. A few people wrote down expressions such as $V = X - GYW$ in which signals values were multiplied (in this case Y and W); this is dimensionally inconsistent and therefore cannot possibly be correct. Some people wrote down the correct initial equations but were not able to solve them. Initially there are three equations and three unknown variables (we assume that the input, X, is known). What you want to do is to eliminate the internal variables V and W by substitution and this will leave one equation that gives Y in terms of X.. One or two people wrote down KCL equations at the nodes of the diagram; this is not valid since there are no currents flowing in the "wires".

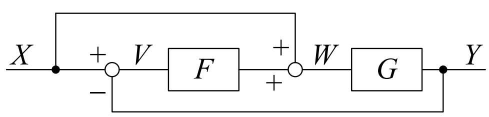


Figure 1.7

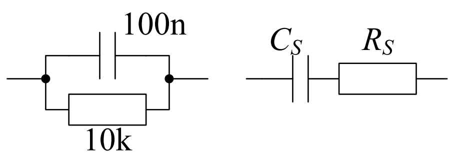


Figure 1.8

g) i) Determine C_S and R_S so that the two networks in Figure 1.8 have the same impedance at $\omega_0 = 2000\,\text{rad/s}$.

ii) Using logarithmic axes for both frequency and impedance sketch a graph showing the impedance magnitude of both networks for the frequency range $20 < \omega < 200000$. [5]

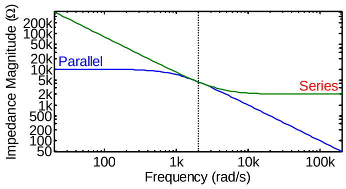
[U] The impedance of the parallel network is $Z_P(j\omega) = \frac{R_P\times\frac{1}{j\omega C_P}}{R_P+\frac{1}{j\omega C_P}} = \frac{R_P}{j\omega R_P C_P+1}$. *For the given component values and at* ω_0 *this is* $Z_P(\omega_0) = \frac{10^4}{j2000\times 10^{-3}+1} = \frac{10^4}{1+2j} = 2-4j\,\mathrm{k}\Omega$.

The impedance of the series network is $Z_S(j\omega) = R_S + \frac{1}{j\omega C_S} = R_S - \frac{1}{\omega C_S}j$. *So therefore, we must have* $R_S = 2\,\mathrm{k}\Omega$ *and* $\frac{1}{\omega_0 C_S} = 4\,\mathrm{k}\Omega$ *from which* $C_S = \frac{1}{2000\times 4000} = 125\,\mathrm{nF}$.

The impedance of the parallel network has an LF asymptote of $R_P = 10\,\mathrm{k}\Omega$ *and an HF asymptote of* $\frac{1}{j\omega C_P} = \frac{10^7}{j\omega}$. *The corner frequency is* $\frac{1}{R_P C_P} = 1000\,\mathrm{rad/s}$.

The impedance of the series network is $Z_S(j\omega) = R_S + \frac{1}{j\omega C_S} = \frac{j\omega R_S C_S+1}{j\omega C_S}$. *This has an LF asymptote of* $\frac{1}{j\omega C_S} = \frac{8\times 10^6}{j\omega}$, *an HF asymptote of* $R_S = 2\,\mathrm{k}\Omega$ *and a corner frequency of* $\frac{1}{R_S C_S} = 4000\,\mathrm{rad/s}$.

Combining all this gives the following graph



Most people got the component values correctly but surprisingly few could draw the graphs; perhaps because it was for an impedance rather than a voltage gain (even though the expression is in both cases one polynomial in $j\omega$ *divided by another). When there is only one corner frequency (as in both these cases) you can draw the graph just by finding the low and high frequency asymptotes. Many people drew graphs that had the impedance of the series circuit tending to zero at low frequencies; the impedance of a passive network involving only capacitors and resistors must always monotonically decrease with frequency. A few people got the formulae for parallel and series circuits interchanged while others wrote dimensionally inconsistent expressions like* $Z_S(j\omega) = \frac{1}{R_S} + \frac{1}{j\omega C_S}$. *Instead of matching the real and imaginary parts of* Z_P *and* Z_S *directly, some people expressed* Z_S *in the form* $Z_S = \frac{j\omega R_S C_S+1}{j\omega C_S}$ *and, in some cases, then tried to match the argument and magnitude; although this is correct, it is much much more complicated. Several people wrote* $\frac{10^4}{1+2j} = \frac{10^4}{3}$; *not only is it invalid to ignore "j" like this, but it also means that you only get one equation rather than two (since the real and imaginary parts of a complex-valued equation must both match). A few gave the impedances in decibels (decibels are reserved for power, voltage or current ratios). It is much easier to match the impedance of the networks by writing the series impedance in the form* $Z_S(j\omega) = R_S + \frac{1}{j\omega C_S}$ *rather than writing it as a single fraction; some people wrote a lot of alebra for this stage. Some people attempted to find a general formula for* R_S *and* C_S *in terms of* R_P *and* C_P. *This is quite possible to do but involves much more algebra than substituting in the known values for* R_P *and* C_P *directly into the expression for* Z_P. *Component values are always real-valued so* $C = 125j\,\mathrm{nF}$ *is unlikely to be a correct solution.*

h) The waveform, $x(t)$, shown in Figure 1.9 is applied to the input, X, of the circuit shown in Figure 1.10. Determine the time constant of the circuit and the amplitude of the transient component of $y(t)$.

Hence draw a dimensioned sketch of the waveform at Y . [5]

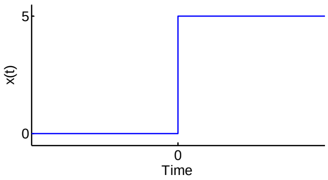


Figure 1.9

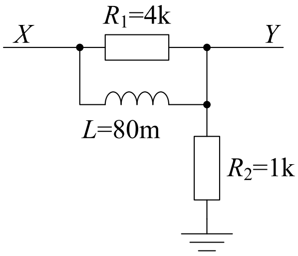
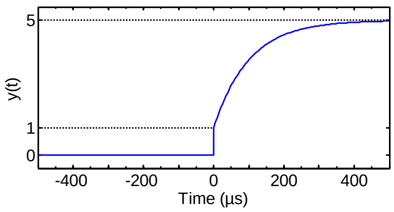


Figure 1.10

[U] The time constant can be determined from the Thevenin resistance of the nework connected across the inductor. By short-circuitiung the input, we find that this is 4k *in parallel with* 1k *which is* $800\,\Omega$*. Hence the time constant is* $\frac{L}{R} = \frac{80}{800} = 0.1\,\text{ms}$*. The steady state output is* $y = x$ *since the inductor acts as a short circuit for DC. At time* $t = 0+$ *there is no current through the inductor and so* $y(0+) = \frac{1}{4+1}x(0+) = 0.2 \times 5 = 1\,\text{V}$*. Hence the transient amplitude is* $y(0+) - y(\infty) = 1 - 5 = -4\,\text{V}$*.*

An alternative method is to determine the transfer function as $H(j\omega) = \frac{R_2}{\frac{j\omega R_1 L}{j\omega L + R_1} + R_2} = \frac{R_2(j\omega L + R_1)}{j\omega L(R_1+R_2) + R_1R_2}$*. From this we find that the DC gain is* $H(j0) = 1$*, the HF gain is* $H(j\infty) = \frac{R_2}{R_1+R_2} = 0.2$ *and the time constant (which equals the reciprocal of the corner frequency) is* $\frac{L(R_1+R_2)}{R_1R_2} = 0.1\,\text{ms}$*. Note that the DC gain and HF gain can also be deduced directly from the circuit by setting the inductor to a short circuit (DC gain) or open circuit (HF gain) respectively.*

The full expression for $y(t)$ *is* $y(t) = \begin{cases} 0 & t < 0 \\ 5 - 4e^{-\frac{t}{\tau}} & t \geq 0 \end{cases}$ *and this is plotted below;* $y(t)$ *jumps up to* 1 *at* $t = 0+$ *and then rises more slowly to its steady state valuee of* 5V*.*



Quite a lot of people correctly said that there was no current through the inductor at $t = 0+$ *but assumed this meant that, at the instant of* $t = 0+$*, it acted as a short circuit and hence that* $y(0+) = x(0+)$ *. In fact, if there is no current flowing through a component, it acts as an* <u>*open*</u> *circuit. Conversely, quite a lot of people said that the steady state output for* $t > 0$ *was* $y_{SS} = 1$ *rather than the actual value of* $y_{SS} = 5$*. Both these assumtions would have been correct if the inductor was replaced by a capacitor. Several people calculated the current through the inductor as* $i(t) = 5\left(1 - e^{\frac{-t}{\tau}}\right)$ mA *but almost all of them then said* $y(t) = R_2 i(t)$ *which ignores the current through* R_1*. There was quite often* *confusion between the steady state and the conditions at $t = 0+$; the steady state is what happens when the transient has died away and, in this case, is* $y_{SS}(t) = \begin{cases} 0 & t < 0 \\ 5 & t \geq 0 \end{cases}$.

---

2. A second order transfer function is given by

$$H(j\omega) = \frac{-G}{\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1}$$

where G, ω_0 and ζ are positive real numbers.

a) Determine the magnitude and phase of $H(j\omega)$ at [4]

i) $\omega = 0$,

ii) $\omega = \omega_0$,

iii) $\omega \gg \omega_0$.

[U] At $\omega = 0$, $H(j\omega) = -G$ which therefore has magnitude G and phase π rad (or, equivalently, $-\pi$ rad).

At $\omega = \omega_0$, $\left(\frac{j\omega}{\omega_0}\right)^2 = -1$ so that $H(j\omega) = \frac{-G}{2j\zeta} = j\frac{G}{2\zeta}$. This has magnitude $\frac{G}{2\zeta}$ and phase $\frac{\pi}{2}$ rad (or $-\frac{3\pi}{2}$ rad).

At $\omega \gg \omega_0$, the $\left(\frac{j\omega}{\omega_0}\right)^2 = -\frac{\omega^2}{\omega_0^2}$ term dominates in the denominator, and so the magnitude tends to $\frac{G\omega_0^2}{\omega^2}$ and the phase tends to 0.

Quite a few people gave the magnitudes but not the phases, perhaps becuase they didn't read the question carefully enough. Not everyone realized that the phase (a.k.a. argument) of a real number can be either 0 or π according to whether the number is positive or negative. Thus $\angle -G = \pi$ and not 0 since $-G$ is negative. Surprisingly many people said $|H(j0)| = -G$; the magnitude of a complex number must always be a non-negative real number (note that the question explicitly states that G is positive). At $\omega = \infty$, the gain is 0 and the phase is indeterminate; however, as $\omega \to \infty$ the phase tends to 0 (or, equivalently, any multiple of 2π) as can be seen above. Over the range $\omega = 0$ to $\omega = \infty$ the phase decreases by a total of π; when plotting the phase response in part (c-ii), it is necessary to add/subtract multiples of 2π onto the phase values to make the phase variation continuous. To ensure this, the phase change beween successive frequencies should always lie within the range $\pm\pi$; thus $\phi = \pi \cdots \frac{\pi}{2} \cdots 0$ or $\phi = -\pi \cdots -\frac{3\pi}{2} \cdots -2\pi$ are acceptable sequences but $\phi = \pi \cdots -\frac{3\pi}{2} \cdots 0$ is not. A few people interpreted $\omega \gg \omega_0$ to mean that $\omega_0 = 1$ which is not the same thing at all. You have to be careful when using the formula $\angle z = \tan^{-1}\frac{\Im(z)}{\Re(z)}$ because this only determines the argument of z to within an arbitrary multiple of π; thus you would get the same result for $\angle(1+j)$ as for $\angle(-1-j)$ whereas in fact their arguments are $+\frac{\pi}{4}$ and $-\frac{3\pi}{4}$ respectively. You can use the signs of $\Re(z)$ and $\Im(z)$ to determie which quadrant of the Argand diagram contains z and hence work out whether you need to add π or not. When $\omega \gg \omega_0$ you need only retain the term with the highest power of $j\omega$ in the denominator, $\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1$; some people correctly neglected the "1" but kept both the other terms.

---

b) If we define $\phi(\omega) = \angle H(j\omega)$, show that $\phi(\omega) = \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega^2 - \omega_0^2}\right)$ and hence show that its derivative at ω_0 equals $\phi'(\omega_0) = \frac{-1}{\zeta\omega_0}$. [6]

[U] Using the formulae $\angle \frac{y}{z} = \angle y - \angle z$ *and* $\angle z = \tan^{-1} \frac{\Im(z)}{\Re(z)}$, *we have*

$$\begin{aligned}
\phi(\omega) &= \angle(-G) - \angle\left(\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\frac{j\omega}{\omega_0} + 1\right) \\
&= \pi - \angle\left(-\omega^2 + j2\zeta\omega_0\omega + \omega_0^2\right) \\
&= \pi - \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2}\right) \\
&= \pi + \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega^2 - \omega_0^2}\right) \\
&= \tan^{-1}\left(\frac{2\zeta\omega_0\omega}{\omega^2 - \omega_0^2}\right)
\end{aligned}$$

where the last line depends on $\tan(\theta + \pi) = \tan(\theta)$ *and the previous line on* $\tan(-\theta) = -\tan(\theta)$. *Alternatively, the* π *term arising from the numerator can be eliminated by initially multiplying numerator and denominator by* -1.

Differentiating, and using the formula $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ *together with the chain rule, we get*

$$\begin{aligned}
\phi'(\omega) \quad &= \quad \frac{1}{1 + \left(\frac{2\zeta\omega_0\omega}{\omega^2 - \omega_0^2}\right)^2} \times 2\zeta\omega_0 \frac{\left(\omega^2 - \omega_0^2\right) - 2\omega \times \omega}{\left(\omega^2 - \omega_0^2\right)^2} \\
= 2\zeta\omega_0 \quad & \frac{-\left(\omega_0^2 + \omega^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + \left(2\zeta\omega_0\omega\right)^2}
\end{aligned}$$

If we now substitute $\omega = \omega_0$ *we get*

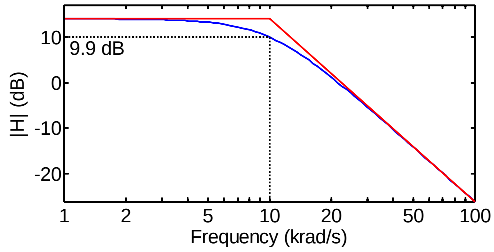
$$\begin{aligned}
\phi'(\omega_0) &= \frac{-4\zeta\omega_0^3}{\left(2\zeta\omega_0\omega\right)^2} \\
&= \frac{-4\zeta\omega_0^3}{4\zeta^2\omega_0^4} = \frac{-1}{\zeta\omega_0}
\end{aligned}$$

Most people got this correct although sometimes after a lot of algebra. A very large number of people quietly ignored the argument of the numerator which is π *rather than* 0; *however since* $\tan()$ *has period* π *they still got the right answer. The argument of a complex fraction can be calculated as the argument of the numerator minus the argument of the denominator; multiplying both by the complex conjugate of the denominator gives the same result but makes the algebra quite a bit worse. Many got the wrong sign for the argument of* $\tan^{-1}()$; *a negative sign arises because the expression is the* <u>*denominator*</u> *of a fraction. A few people wrongly assumed that the real and imaginary parts of* $\frac{1}{a+jb}$ *were* $\frac{1}{a}$ *and* $\frac{1}{b}$ *respectively; if you want to find the real and imaginary parts, you need to multiply numerator and denominator by* $a - jb$ *although, as noted above, you do not need to do this for this question. Quite a few people didn't use the chain rule and just said* $\phi'(\omega) = \frac{1}{1+\left(\frac{2\zeta\omega_0\omega}{\omega^2-\omega_0^2}\right)^2}$ *which is much simpler but wrong. Most people knew the derivative of* $\tan^{-1} x$ *but a few did not; it is possible to derive it as follows (or in many other ways):* $x = \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \frac{dx}{d\theta} = \frac{(\cos\theta\times\cos\theta)-(\sin\theta\times-\sin\theta)}{\cos^2\theta} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = 1 + x^2$.

c) Suppose that $G = 5$, $\zeta = 0.8$ and $\omega_0 = 10^4$ rad/s.

i) Sketch a dimensioned graph of $|H(j\omega)|$ in decibels using a logarithmic frequency axis. Your graph should include a sketch of the true magnitude response in addition to the high and low frequency asymptotes. [3]

[B] The LF asyptote has a gain of $5 = 14\,\text{dB}$. The HF asymptote has a gradient of $-40\,\text{dB}$ per decade and meets the LF asymptote at the corner frequency, ω_0. From part a) the gain at ω_0 is $\frac{G}{2\zeta} = \frac{5}{1.6} = 3.12 = 9.9\,\text{dB}$.

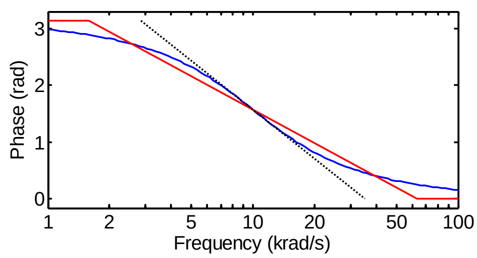


A few people thought that a low frequency gain of -5 corresponded to $-14\,\text{dB}$; you need to take the magnitude of the gain before converting to decibels. Surprisingly many people drew the graph with a peak at, or near, ω_0 even though they correctly gave the expression $|H(j\omega_0)| = \frac{G}{2\zeta}$ in part (a). Some people drew a narrow peak going downwards at $\omega = \omega_0$; this is wrong on two counts (a) if there is a peak, it is always in the opposite direction to the asymptote gradient change (i.e. upwards in this case) and (b) there is no peak at all if $|\zeta| \leq 0.7$. The word "dimensioned" in the question means that you need some values on the axes; quite often these were entirely missing.

---

ii) Sketch a dimensioned graph of $\angle H(j\omega)$ using a linear phase axis in radians and a logarithmic frequency axis. [3]

[B] From part a) the LF and HF asymptotes are π and 0 respectively. The standard 3-line approximation changes from π to 0 over $\pm\zeta$ decades, i.e. $\omega_0 \times \begin{bmatrix} 10^{-\zeta} & 10^{\zeta} \end{bmatrix} = 10^4 \times \begin{bmatrix} 0.158 & 6.31 \end{bmatrix} = \begin{bmatrix} 1.58 & 63.1 \end{bmatrix}$ krad/s. This is shown as the solid line below. The gradient of the standard 3-line approximation is $\frac{-\pi}{2\zeta} = \frac{-1.57}{\zeta} = -1.96$.



[The following is not required in the answer but included for interest]: If $x = \log_{10}\omega = \frac{1}{\ln 10}\ln\omega$, then $\frac{d\phi}{dx} = \frac{d\phi}{d\omega} \times \frac{d\omega}{dx} = \ln(10)\omega\phi'(\omega)$.

The gradient of the phase response curve at $\omega = \omega_0$ is therefore equal to $\frac{-\ln 10}{\zeta} = \frac{-2.3}{\zeta} = -2.88$ which is inversely proportional to ζ. This is shown as the dashed line above and cross the LF and HF asymptotes at $\omega_0 \pm 0.68\zeta$ decades.

The low and high frequency asymptotes of a phase response are always horizontal. Some drew the gradient as positive because they had calculated $\phi(0) = -\pi$ and $\phi(\infty) = 0$ in part (a) and didn't notice that this made the value of $\phi(\omega_0)$ incorrect ($-\frac{\pi}{2}$ instead of $+\frac{\pi}{2}$) See the note in part (a) above about adding/subtracting 2π to avoid such problems. Instead of showing the phase trasition extending over $\pm\zeta$ decades, many people had it extending over ± 1 decades (as is correct for a linear factor).

---

d) Fig. 2.1 shows the circuit diagram of a filter circuit. Assuming the opamp to be ideal, use nodal analysis to show that the frequency response of the filter is given by

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-R_2}{R_1R_2R_3C_1C_2(j\omega)^2 + (R_1R_2 + R_1R_3 + R_2R_3)C_1 j\omega + R_1}.$$

[6]

[U] We do KCL at node V and at the $-$ve op-amp input (which is a virtual earth) to obtain

$$\begin{aligned} \frac{V-X}{R_1} + j\omega C_2 V + \frac{V-Y}{R_2} + \frac{V}{R_3} &= 0 \\ \frac{-V}{R_3} - j\omega C_1 Y &= 0 \end{aligned}$$

We would like to eliminate V between these two equations. We can rearrange them to obtain

$$\begin{aligned} -\frac{1}{R_2}Y + \left(\frac{1}{R_1} + j\omega C_2 + \frac{1}{R_2} + \frac{1}{R_3}\right)V &= \frac{1}{R_1}X \\ V &= -j\omega R_3 C_1 Y \end{aligned}$$

Multiplying the first equation by R_1R_2 and then substituting for $V = -j\omega R_3 C_1 Y$ gives

$$-R_1 Y - \left(R_2 + j\omega R_1 R_2 C_2 + R_1 + \frac{R_1R_2}{R_3}\right) j\omega R_3 C_1 Y = R_2 X$$

from which

$$\frac{Y}{X} = \frac{-R_2}{R_1R_2R_3C_1C_2(j\omega)^2 + (R_1R_2 + R_1R_3 + R_2R_3)C_1 j\omega + R_1}$$

Most people got this right although in some cases they used a very great deal of algebra. Not everyone realized that the negative input of the opamp is at 0 volts; some included its votage as an additional unknown which makes the problem insoluble. Quite a lot of people omitted the term $\frac{V}{R_3}$ from the KCL equation at V; although there is no current into the opamp input, this does not mean there is no current through R_3 because C_1 is also connected to the same node. Some people included invalid, KCL equations by summing currents at node X and/or node Y; the first is no good because you do not know the current supplied to the input at X and the second is no good becuase you do not know the current supplied or drawn by the opamp output. Many people substituted separately for each of the four occurrences of V in the first equation given above; it is much easier if you collect all the terms in V together before substituting. A few people took the impedance of the capacitor to be $j\omega C$ (or in some cases just C) rather than $\frac{1}{j\omega C}$; this is quite an easy mistake to make. Several people used the potential divider formula to say $V = \frac{\frac{1}{j\omega C_2}}{R_1 + \frac{1}{j\omega C_2}}$ which is incorrect; you cannot use the potential divider formula when any other components that might draw current are connected to its mid point (in this case R_2 and R_3 are both connected to node V). Quite a few people solved the problem in terms of $Z_1 = \frac{1}{j\omega C_1}$ and $Z_2 = \frac{1}{j\omega C_2}$ and then substituted for them right at the end; this gives the correct answer and makes it easier to avoid dimensional inconsistencies but it involves quite a bit more algebra.

---

e) Find expressions for G, ω_0 and ζ in terms of the component values when the frequency response of the filter is expressed in the form given for $H(j\omega)$ above. [4]

[U] By dividing the numerator and denominator by R_1 and matching coefficients, we obtain

$$G = \frac{R_2}{R_1} \tag{2.1}$$

$$\omega_0 = \frac{1}{\sqrt{R_2R_3C_1C_2}} \tag{2.2}$$

$$\frac{2\zeta}{\omega_0} = (R_1R_2 + R_1R_3 + R_2R_3)\frac{C_1}{R_1} \tag{2.3}$$

$$\Rightarrow \zeta = \frac{(R_1R_2 + R_1R_3 + R_2R_3)C_1}{2R_1\sqrt{R_2R_3C_1C_2}}$$

Most people got this right. Some omitted to divide by R_1 and so obtained, for example, $G = R_2$ which is dimensionally inconsistent. The gain of any voltage-in to voltage-out amplifier circuit is dimensionless; it cannot have the dimensions of ohms. A few people expressed ζ as $\zeta = \frac{(R_1R_2+R_1R_3+R_2R_3)\omega_0 C_1}{2R_1}$ which is mathematically correct but, since it involves ω_0, is not what the question asked for.

---

f) If $R_2 = 60\,\text{k}\Omega$ and $R_3 = \frac{50}{3}\,\text{k}\Omega$ determine values for R_1, C_1 and C_2 such that $G = 5$, $\zeta = 0.8$ and $\omega_0 = 10^4\,\text{rad/s}$. [4]

[U] From 2.1 we get $R_1 = \frac{R_2}{G} = \frac{60}{5} = 12\,\mathrm{k}\Omega$.

Now from 2.3 we can write

$$\begin{aligned} C_1 &= \frac{2\zeta R_1}{\omega_0 \left(R_1R_2 + R_1R_3 + R_2R_3\right)} \\ &= \frac{2\times 0.8\times 12\times 10^3}{10^4\times\left(12\times 60 + 12\times\frac{50}{3} + 60\times\frac{50}{3}\right)\times 10^6} \\ &= \frac{19.2\times 10^{-7}}{720+200+1000} = 10^{-9} = 1\,\mathrm{nF} \end{aligned}$$

Finally from 2.2, we have

$$\begin{aligned} C_2 &= \frac{1}{\omega_0^2 R_2 R_3 C_1} \\ &= \frac{1}{10^8\times 60\times\frac{50}{3}\times 10^6 C_1} = \frac{10^{-17}}{C_1} \\ &= 10^{-8} = 10\,\mathrm{nF} \end{aligned}$$

Most people got R_1 correct but quite a few failed to determine C_1 and C_2. If you calculate the components in the order given above so that at each stage you use an equation that contains only one unknown. This avoids having to solve any non-linear simultaneus equations. The expression for $\frac{2\zeta}{\omega_0}$ is more convenient to use than the expression for ζ because the former involves only C_1 whereas the latter involves both C_1 and C_2. Most people however used the expressions for ζ and for ω_0 and then solved the simultaneous equations. Quite a common mistake was to forget about the "k" in $60\,\mathrm{k}\Omega$ which resulted in missing factors of 10^3; if there are capacitors or inductors in a circuit, it is very high risk to work in $\mathrm{k}\Omega$ rather than Ω.

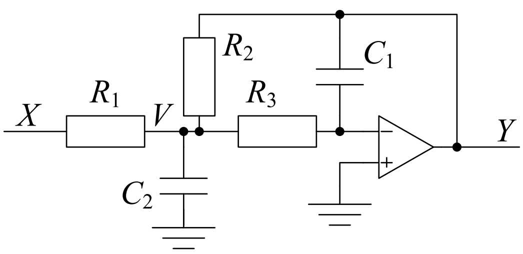


Figure 2.1

3. The circuit of Fig. 3.1 shows a transmission line of length L driven by a sinusoidal voltage source $v_S(t)$ through a resistor, R_S. The characteristic impedance and propagation velocity of the line are Z_0 and u respectively. The phasor corresponding to the waveform $v_S(t)$ is written V_S and similarly for other waveforms.

The voltage and current waveforms at a distance x from the source are given respectively by

$$\begin{aligned} v_x(t) &= f_x(t) + g_x(t) \\ i_x(t) &= Z_0^{-1}\left(f_x(t) - g_x(t)\right) \end{aligned}$$

where $f_x(t) = f_0\left(t - u^{-1}x\right)$ and $g_x(t) = g_0\left(t + u^{-1}x\right)$ are the forward and backward waves at a distance x from the source.

a) Show that if $f_0(t) = A\cos(\omega t + \phi)$ then the phasors F_x and F_0 satisfy

$$F_x = F_0 e^{-j\omega u^{-1}x}.$$

Determine a similar expression relating G_x and G_0. [5]

You may assume without proof that the phasor corresponding to $A\cos(\omega t + \psi)$ is $Ae^{j\psi}$.

[B] If $f_0(t) = A\cos(\omega t + \phi)$, then $F_0 = Ae^{j\phi}$ from the formula given in the question.

At point x on the line,

$$\begin{aligned} f_x(t) &= f_0\left(t - u^{-1}x\right) \\ &= A\cos\left(\omega\left(t - u^{-1}x\right) + \phi\right) \\ &= A\cos\left(\omega t + \left(\phi - \omega u^{-1}x\right)\right). \end{aligned}$$

The phasor F_x is therefore given by $Ae^{j\left(\phi - \omega u^{-1}x\right)} = Ae^{j\phi}e^{-j\omega u^{-1}x} = F_0 e^{-j\omega u^{-1}x}$.

In the same way, if $g_0(t) = A\cos(\omega t + \phi)$, then $g_x(t) = A\cos\left(\omega t + \left(\phi + \omega u^{-1}x\right)\right)$. It follows that $G_x = G_0 e^{+j\omega u^{-1}x}$.

Not everyone was able to write down the correct expression for $f_0\left(t - u^{-1}x\right)$ which can be obtained by replacing every occurence of "t" in $f_0(t) = A\cos(\omega t + \phi)$ by "$\left(t - u^{-1}x\right)$". Some people mixed time waveforms and phasors in the same equation such as, for example, $f_0(t) = Ae^{j\phi}$; this equation makes no sense since (a) the left side is a function of time and the right side is not and (b) the left side is real-valued while the right side is complex. When using phasors, equations and expressions either involve time (and use lower-case symbols for signals) or else they are complex-valued (and use upper-case symbols) but never both at once. Some people appealed to the "time-shift" property of Fourier transforms which pretty much amounts to assuming what you are asked to prove. Most people gave the correct expression for G_x although a few had a negative sign in the exponent.

---

b) Use the load equation $V_L = I_L R_L$ to show that G_0 can be written in the form $G_0 = \rho_L e^{j\theta} F_0$ and determine expressions for the real-valued constants ρ_L and θ. [5]

[B+U] From ohm's law,

$$
\begin{aligned}
V_L &= R_L I_L \\
F_L + G_L &= R_L Z_0^{-1}(F_L - G_L) \\
G_L\left(R_L Z_0^{-1}+1\right) &= F_L\left(R_L Z_0^{-1}-1\right) \\
G_L &= \frac{R_L - Z_0}{R_L + Z_0} F_L = \frac{R_L - Z_0}{R_L + Z_0} F_0 e^{-j\omega u^{-1}L} \\
G_0 &= G_L e^{-j\omega u^{-1}L} = \frac{R_L - Z_0}{R_L + Z_0} F_0 e^{-j2\omega u^{-1}L}
\end{aligned}
$$

Thus $\rho_L = \frac{R_L - Z_0}{R_L + Z_0}$ *and* $\theta = -2\omega u^{-1}L$.

Quite a few people wrote "x" instead of "L" throughtout the derivation. The load equation, $V_L = R_L I_L$, *is only valid at the load end of the line where* $x = L$, *so writing equations involving a general "x" such as* $V_x = R_L I_x$, $V_L = F_x + G_x$ *or* $G_x = \rho_L F_x$ *is incorrect. The algebra is easier if, as above, you wait until the end before substituting* $F_L = F_0 e^{-j\omega u^{-1}L}$ *and* $G_L = G_0 e^{+j\omega u^{-1}L}$; *quite a few people made the substitution early on and then had to manipulate equations that included exponentials with easily forgotten signs. Some people wrote expressions like* $F_L + G_0$ *which makes little sense because it adds together the forward and backward voltages at different points on the line.*

---

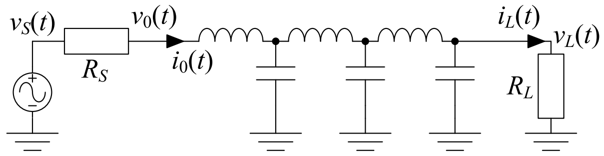
c) By applying Kirchoff's current law at the point marked $v_0(t)$ in Fig. 3.1, show that F_0 may be expressed as $F_0 = \tau_S V_S + \rho_S G_0$ and determine expressions for the real-valued constants τ_S and ρ_S. [5]

[B] KCL at point V_0 *gives*

$$
\begin{aligned}
\frac{V_S - V_0}{R_S} &= I_0 \\
V_S - (F_0 + G_0) &= R_s Z_0^{-1}(F_0 - G_0) \\
F_0\left(R_s Z_0^{-1}+1\right) &= V_S + \left(R_s Z_0^{-1}-1\right) G_0 \\
F_0 &= \frac{Z_0}{R_S + Z_0} V_S + \frac{R_s - Z_0}{R_S + Z_0} G_0
\end{aligned}
$$

Thus $\tau_S = \frac{Z_0}{R_S + Z_0}$ *and* $\rho_S = \frac{R_s - Z_0}{R_S + Z_0}$.

Several people had a sign error in the original equation and wrote $\frac{V_0 - V_S}{R_S} = I_0$. *A few people decomosed* V_S *as* $V_S = F_S + G_S$ *which doesn't make any sense; the "x" in* F_x *refers to a point on the line that is a distance x from the source and "S" is a node name rather than a distance. Quite a lot of people wrongly said that* $I_0 = \frac{V_0}{R_L}$; *this implicitly assumes that* $i_0(t) = i_L(t)$ *and that* $v_0(t) = v_L(t)$ *but it is a fundamental property of transmission lines that the voltage and current are different at different points on the line. Another common mistake was to say that* $I_0 = \frac{V_0 - V_L}{Z_0}$ *as if the transmission line had a series resitance of* Z_0. *This is completely untrue; the characteristic impedance,* Z_0, *is not an actual resistance anywhere in the circuit but arises in the formula for* $i_x(t)$ *that is given in the question. The properties of the transmission line arise from its distributed inductance and capacitance; if we show these explicitly on the diagram, it becomes clear why* $v_0(t)$ *and* $v_L(t)$ *are not the same.*



d) Eliminate G_0 between the answers to parts b) and c) to obtain an expression for F_0 in terms of V_S. [4]

[U] Substituting the expression for G_0 into the answer for c) gives

$$\begin{aligned} F_0 &= \tau_S V_S + \rho_S G_0 \\ &= \tau_S V_S + \rho_S \times \rho_L F_0 e^{j\theta} \\ F_0\left(1 - \rho_S \rho_L e^{j\theta}\right) &= \tau_S V_S \\ F_0 &= \frac{\tau_S}{1 - \rho_S \rho_L e^{j\theta}} V_S \end{aligned}$$

Most people got this correct.

e) Suppose that $R_S = 25\,\Omega$, $R_L = 400\,\Omega$, $Z_0 = 100\,\Omega$, $L = 10\,\text{m}$, $u = 1.5 \times 10^8\,\text{m/s}$, $V_S = 10j$ and $\omega = 6 \times 10^7\,\text{rad/s}$.

Determine the phasors V_0 and I_0. [6]

[U] From part b), $\rho_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{300}{500} = 0.6$ and $\theta = -2\omega u^{-1} L = -8\,\text{rad}$ which means that $e^{j\theta} = -0.146 - 0.989j$. From c), $\tau_S = \frac{Z_0}{R_S + Z_0} = \frac{100}{25+100} = 0.8$ and $\rho_S = \frac{R_s - Z_0}{R_S + Z_0} = -0.6$. It follows that

$$\begin{aligned} F_0 &= \frac{\tau_S}{1 - \rho_S \rho_L e^{-j2\omega u^{-1} L}} V_S \\ &= \frac{8j}{1 + 0.36(-0.146 - 0.989j)} \\ &= \frac{8j}{1 - 0.0524 - 0.356j} \\ &= \frac{8j}{0.948 - 0.356j} \\ &= -2.78 + 7.4j = 7.91\angle 1.93 \end{aligned}$$

from which

$$\begin{aligned} G_0 &= \rho_L e^{j\theta} F_0 \\ &= 0.6(-0.146 - 0.989j)(-2.78 + 7.4j) \\ &= 4.63 + 1j = 4.74\angle 0.21 \end{aligned}$$

So now we have

$$\begin{aligned} V_0 &= F_0 + G_0 = 1.85 + 8.4j = 8.60\angle 1.35 \\ I_0 &= Z_0^{-1}(F_0 - G_0) = -74 + 64j\,\text{mA} = 97.8\angle 2.43\,\text{mA} \end{aligned}$$

Alternatively, we can just use ohms law to get I_0:

$$I_0 = \frac{V_s - V_0}{R_s} = \frac{10j - (1.85 + 8.4j)}{25} = -74 + 64j\,\text{mA}.$$

Many people wrote $V_0 = \tau_S V_S$ which is not true. If we want to, we can express V_0 in terms of V_S by combining two of the results derived in this question: $V_0 = F_0 + G_0 = F_0\left(1 + \rho_L e^{j\theta}\right) = \frac{\tau_S}{1-\rho_S\rho_L e^{j\theta}}\left(1 + \rho_L e^{j\theta}\right) V_S = \left(\frac{1+\rho_S^{-1}}{1-\rho_S\rho_L e^{j\theta}} - \rho_S^{-1}\right)\tau_S V_S$. Several people obtained complicated arithmetic expressions for V_0 and I_0 but lost marks because they did not evaluate them. If you are asked to determine a complex value, your answer must be in one of the standard forms: either real+imaginary or magnitude+argument.

---

f) Calculate the complex power supplied by V_S and the average power absorbed by R_S. Hence deduce the average power absorbed by R_L. [5]

[U] The complex power supplied by V_S is

$$\frac{1}{2}V_S I_0^* = 0.5\,(10j)\,(-74 + 64j) = 320 - 371j\,\text{mVA}$$

The average power absorbed by R_S is

$$\begin{aligned}\frac{1}{2}|I_0|^2 R_S &= 0.5 \times 0.0979^2 \times 25 \\ &= 120\,\text{mW}\end{aligned}$$

The average power absorbed by R_L must therefore be $320 - 120 = 200\,\text{mW}$. Alternatively, the complex power absorbed by the line+load combination is

$$\frac{1}{2}V_0 I_0^* = 0.5\,(1.85 + 8.4j)\,(-74 + 64j) = 200 - 371j\,\text{mVA}$$

and the real part of this, $200\,\text{mW}$, gives the average power absorbed (which must all be in R_L).

Some said the "current along the line" was $I = \frac{V_S}{R_S + Z_0 + R_L}$; however there is no such thing as the "current along the line" since the current is different in different places (see comment on part (c) above). The current through R_S is though the same as I_0. Note that it is not true that the power entering the line is $\frac{1}{2}|I_0|^2 Z_0$ as some thought. Instead, the net power entering the line is the difference between the powers carried by the forward and backwards waves: $\frac{|F_0|^2}{2Z_0} - \frac{|G_0|^2}{2Z_0} = 312 - 112 = 200\,\text{mW}$. The complex power absorbed by R_L (or for that matter any resistor) must be real-valued; the imaginary part of $\frac{1}{2}V_0 I_0^*$ (which equals the imaginary part of $\frac{1}{2}V_S I_0^*$) is absorbed by the transmission line capacitance and inductance.

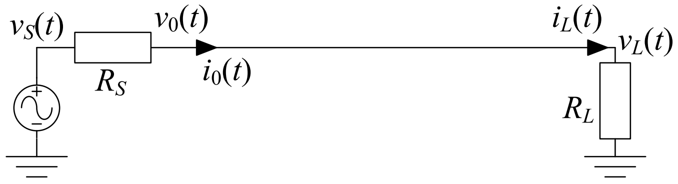


Figure 3.1