IMPERIAL COLLEGE LONDON

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING
EXAMINATIONS 2017

EEE/EIE PART I: MEng, BEng and ACGI

ANALYSIS OF CIRCUITS

Tuesday, 6 June 10:00 am

Time allowed: 2:00 hours

There are THREE questions on this paper.

Answer ALL questions.
Q1 carries 40% of the marks. Questions 2 and 3 carry equal marks (30% each).

Any special instructions for invigilators and information for candidates are on page 1.

Examiners responsible First Marker(s) : D.M. Brookes
Second Marker(s) : P. Georgiou

ANALYSIS OF CIRCUITS

Information for Candidates:

- Numerical answers must be given as fully evaluated decimal values and not as unevaluated arithmetic expressions.

Notation

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [4]

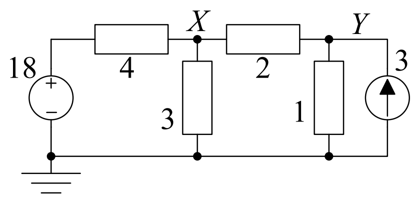


Figure 1.1

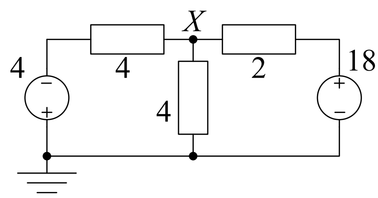


Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [4]

c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [4]

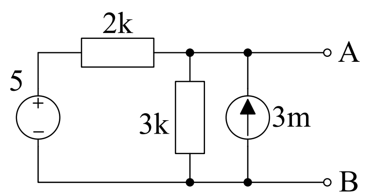


Figure 1.3

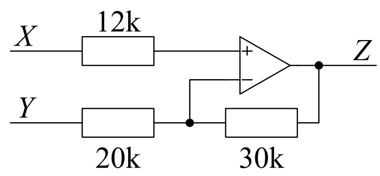


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [4]

e) The diode in the circuit of Figure 1.5 has a forward voltage of 0.7 V when conducting but is otherwise ideal. Determine the output voltage, Y, when
(i) $X = 1\,\mathrm{V}$,
(ii) $X = 5\,\mathrm{V}$
(iii) $X = -5\,\mathrm{V}$. [5]

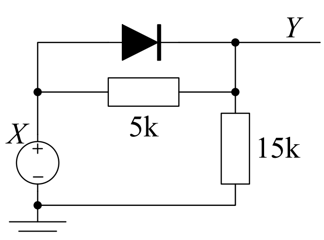


Figure 1.5

f) i) The diagram of Figure 1.6 shows an AC source with r.m.s. voltage 230 V driving a load with impedance $50+25j\,\Omega$ through a line with impedance $2\,\Omega$.

Determine the complex power, given by $S = \widetilde{V} \times \widetilde{I}^*$, absorbed by the load and the complex power absorbed by the 2Ω resistor. [4]

ii) A capacitor with impedance $-200j$ is now connected across the load, as indicated in Figure 1.7. Determine the complex power absorbed by the load and the complex power absorbed by the 2Ω resistor. [4]

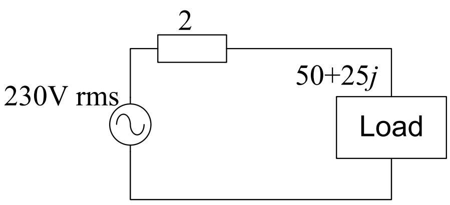


Figure 1.6

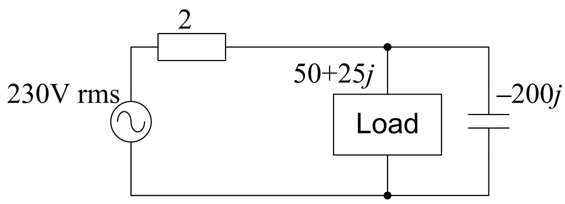


Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circle represents an adder/subtractor; its three inputs have the signs indicated on the diagram and its output is V. [4]

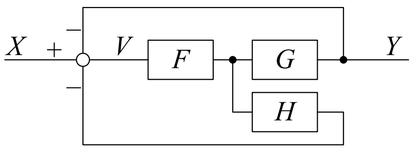


Figure 1.8

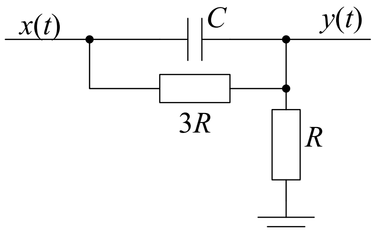


Figure 1.9

h) The input voltage in Figure 1.9 is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 8\,\text{V} & t \geq 0. \end{cases}$$

i) Determine the time constant of the circuit. [2]

ii) Determine an expression for $y(t)$ for $t > 0$. [5]

2. The frequency response of a circuit is given by

$$H(j\omega) = \frac{aj\omega}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where a, ζ and ω_0 are real numbers.

a) i) By dividing the numerator and denominator of $H(j\omega)$ by $j\omega$ and then multiplying the resultant expression by its complex conjugate, show that $|H(j\omega)|^2 = \dfrac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2}$. [3]

ii) Explain why the maximum value of $|H(j\omega)|^2$ occurs when the quantity $\left(\omega - \frac{\omega_0^2}{\omega}\right)$ equals zero. Hence show that the maximum occurs at $\omega = \omega_0$ and determine $|H(j\omega_0)|^2$. [2]

iii) Find expressions for the two positive values of ω for which $|H(j\omega)|^2 = \dfrac{a^2}{8\zeta^2\omega_0^2}$ and determine a simplified expression for the difference between them. [4]

b) Suppose now that $a = 5000\,\mathrm{s}^{-1}$, $\zeta = 0.1$ and $\omega_0 = 5000\,\mathrm{rad/s}$.

i) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

ii) Draw a dimensioned sketch showing the high and low frequency asymptotes as well as the true magnitude response, $|H(j\omega)|$. Indicate on your graph in dB the peak value of $|H(j\omega)|$ and the value of the asymptotes at their point of intersection. [5]

iii) Draw a dimensioned sketch of the straight-line approximation to the phase response, $\angle H(j\omega)$. You may assume without proof that the gradient of the approximation at ω_0 is equal to $-0.5\pi\zeta^{-1}$ radians per decade where "decade" means a factor of 10 in frequency. [4]

c) i) Show that the frequency response, $\frac{Y(j\omega)}{X(j\omega)}$ of the circuit shown in Figure 2.1 is given by [5]

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega R_2 C}{(j\omega)^2 R_1 R_2 C^2 + 2j\omega R_1 C + 1}.$$

ii) Determine simplified expressions for a, ζ and ω_0 so that the expression given in part c)i) equals that given above for $H(j\omega)$. [3]

iii) Given that $C = 10\,\mathrm{nF}$, determine the values of R_1 and R_2 so that $\omega_0 = 5000\,\mathrm{rad/s}$ and $\zeta = 0.1$. [2]

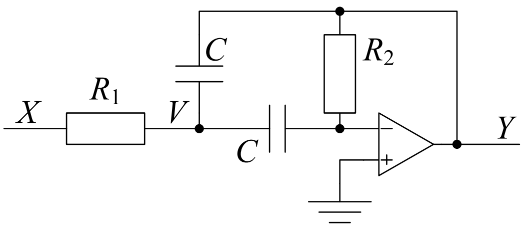


Figure 2.1

3. Figure 3.1 shows a shows a transmission line of length $L = 10\,\mathrm{m}$ whose characteristic impedance is $Z_0 = 120\,\Omega$ and whose propagation velocity is $u = 2 \times 10^8\,\mathrm{m/s}$. Distance along the line is denoted by x and the two points $x = 0$ and $x = L$ are marked in the figure.

At a point x on the line, the line voltage and current are given by $v_x(t) = f_x(t) + g_x(t)$ and $i_x(t) = Z_0^{-1}(f_x(t) - g_x(t))$ where $f_x(t) = f_0(t - u^{-1}x)$ and $g_x(t) = g_0(t + u^{-1}x)$ are the forward and backward waves respectively.

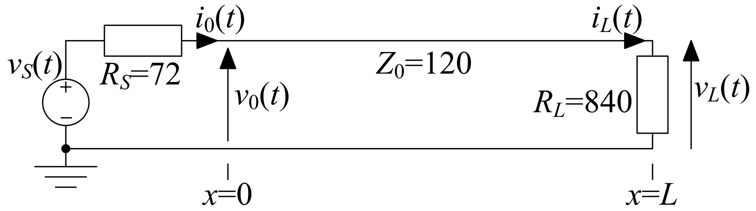


Figure 3.1

a) i) At the position $x = L$, the backward wave is given by $g_L(t) = \rho_L f_L(t)$ where $\rho_L = 0.75$ is the reflection coefficient at $x = L$.

Show that $g_0(t) = \rho_L f_0(t - 2u^{-1}L)$. [3]

ii) At $x = 0$, show that $v_s(t) = v_0(t) + R_S i_0(t)$. Hence show that $f_0(t)$ can be written in the form $f_0(t) = \tau_0 v_s(t) + \rho_0 g_0(t)$ and determine the numerical values of τ_0 and ρ_0. [6]

iii) By combining the results of parts i) and ii) show that

$$f_0(t) = \tau_0 v_s(t) + \rho_0 \rho_L f_0(t - 2u^{-1}L).$$

Hence prove, by using induction or otherwise, that

$$f_0(t) = \sum_{n=0}^{\infty} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right).$$ [6]

b) If the source is a 30 ns pulse given by

$$v_s(t) = \begin{cases} 25.6\,\mathrm{V} & \text{for } 0 \le t \le 30\,\mathrm{ns} \\ 0 & \text{otherwise} \end{cases},$$

draw a dimensioned sketch of the waveform $v_x(t)$ on the line at the point $x = 8\,\mathrm{m}$ for the time interval $0 \le t \le 150\,\mathrm{ns}$. Give the times of all discontinuities and the values of all horizontal portions of the waveform. [6]

c) Now assume that all voltages and currents are sinusoidal with angular frequency ω. The uppercase letter, V_x, denotes the phasor corresponding to $v_x(t)$.

i) The waveform $f_0(t) = A\cos(\omega t + \theta)$ is represented by the phasor $F_0 = Ae^{j\theta}$. Show that $F_x = F_0 e^{-jkx}$ where $k = u^{-1}\omega$. [3]

ii) By converting the first equation given in part a)iii) into phasor form, determine an expression for F_0 in terms of V_s. [3]

iii) Determine an expression for V_x in terms of V_s. [3]

ANALYSIS OF CIRCUITS

**** Solutions 2017 ****

Information for Candidates:

- Numerical answers must be given as fully evaluated decimal values and not as unevaluated arithmetic expressions.

Notation

The following notation is used in this paper:

1. The voltage waveform at node X in a circuit is denoted by $x(t)$, the phasor voltage by X and the root-mean-square (or RMS) phasor voltage by $\widetilde{X} = \frac{X}{\sqrt{2}}$. The complex conjugate of X is X^*.
2. Component and source values in a circuit are normally given in Ohms, Farads, Henrys, Volts or Amps with the unit symbol omitted. Where an imaginary number is specified, it represents the complex impedance or phasor value.
3. Times are given in seconds unless otherwise stated.
4. Unless otherwise indicated, frequency response graphs should use a linear axis for phase and logarithmic axes for frequency and magnitude.
5. The real and imaginary parts of a complex number, X, are written $\Re(X)$ and $\Im(X)$ respectively.

Key: B=bookwork, U=unseen example

1. a) Using nodal analysis, calculate the voltages at nodes X and Y of Figure 1.1. [4]

[U] KCL at node X gives

$$\begin{aligned}\frac{X-18}{4}+\frac{X}{3}+\frac{X-Y}{2} &= 0\\ \Rightarrow 3X-54+4X+6X-6Y &= 0\\ \Rightarrow 13X-6Y &= 54\end{aligned}$$

KCL at node Y gives

$$\begin{aligned}\frac{Y-X}{2}+\frac{Y}{1}-3 &= 0\\ \Rightarrow -X+3Y &= 6\end{aligned}$$

Solving these simultaneous equations gives

$$X=6, \qquad Y=4.$$

Most people got this right. The most common mistake was to multiply one of the terms in an equation by the wrong factor when removing the fractions (e.g. the final term in the top equation above sometimes became $3X-3Y$ or even $X-Y$ instead of $6X-6Y$).

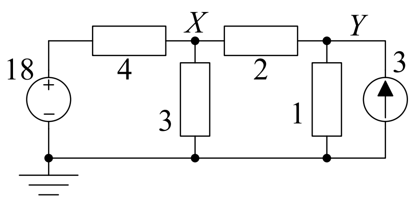


Figure 1.1

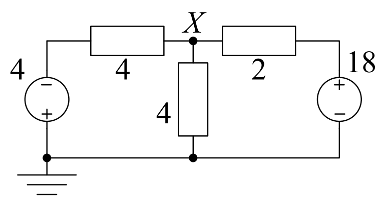


Figure 1.2

b) Use the principle of superposition to find the voltage X in Figure 1.2. [4]

[U] If we short circuit the $18\,\text{V}$ source, the $2\,\Omega$ and $4\,\Omega$ resistors are inparallel and are equivalent to a $\frac{2\times 4}{2+4}=\frac{8}{6}=1.333\,\Omega$ resistor. The circuit is now a potential divider and the voltage at X is given by $X_1=\frac{1.333}{4+1.333}\times -4=\frac{1.333}{5.333}\times -4=-1\,\text{V}$.

If we now short circuit the $4\,\text{V}$ voltage source, the two $4\,\Omega$ resistors are in parallel and equal $2\,\Omega$. The voltage at X is then $X_2=\frac{2}{2+2}\times 18=\frac{1}{2}\times 18=9\,\text{V}$.

By superposition, the total voltage is therefore $X=X_1+X_2=-1+9=8\,\text{V}$.

Most people got this question right except for the occasional arithmetic error. Learning how to use the simultaneous equation function in the calculator is *recommended; it saves a lot of time and reduces arithmetic errors. You do however need to be careful to enter the information in the correct form. Quite a few people used the wrong section of the potential divider, e.g. writing* $X_1 = \frac{4}{4+1.333} \times -4$*; if you want to calculate the voltage at node X, you need to find the voltage across the resistor between X and ground.*

---

c) Draw the Thévenin equivalent circuit of the two-terminal network in Figure 1.3 and find the values of its components. [4]

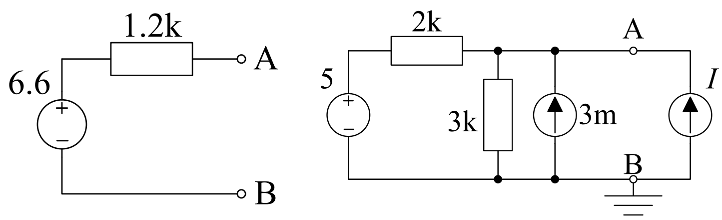
[U] We can find the Thévenin resistance by short-circuiting the voltage source and open-circuiting the current source. This leaves two resistors in parallel with an equivalent resistance of $R_{Thev} = \frac{2\times 3}{2+3} = 1.2\,\text{k}\Omega$.

We can find the open circuit voltage by nodal analysis or by superposition.

(i) Using nodal analysis (and grounding node B): $\frac{A-5}{2} + \frac{A}{3} - 3 = 0$ *from which* $V_{Thev} = A = \frac{33}{5} = 6.6\,\text{V}$.

(ii) By superposition: $V_{5\text{V}} = \frac{3}{3+2} \times 5 = 3\,\text{V}$ *and* $V_{3\text{m}} = \frac{2\times 3}{2+3} \times 3 = 3.6\,\text{V}$ *from which* $V_{Thev} = 3 + 3.6 = 6.6\,\text{V}$.

Either way, we get the diagram on the left below. Alternatively we can ground node B and append a current source, I, as shown in the rightmost diagram below. Now doing KCL at node A gives $\frac{A-5}{2} + \frac{A}{3} - 3 - I = 0$ *from which* $A = 6.6 + 1.2I$ *which gives* V_{Thev} *and* R_{Thev} *directly.*



Another, trixier, method of solving this question is to convert the $5\,\text{V}$*and* $2\,\text{k}\Omega$ *into their Norton equivalent:* $2\,\text{k}\Omega$ *in parallel with* $\frac{5}{2} = 2.5\,\text{mA}$ *upwards. Now combine the parallel resistors and the parallel current sources to get* $1.2\,\text{k}\Omega$ *in parallel with* $5.5\,\text{mA}$*. Finally, convert back from Norton to Thévenin to give* $1.2\,\text{k}\Omega$ *and* $1.2 \times 5.5 = 6.6\,\text{V}$.

Several people ignored the current source when calculating the open-circuit voltage. A few people calculated the component values but did not draw the circuit despite the first word of the question being "Draw"; a few people did draw the circuit but put the resister and voltage source in parallel rather than in series.

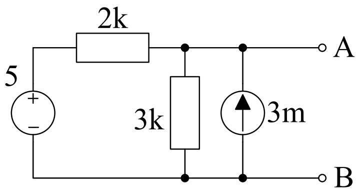


Figure 1.3

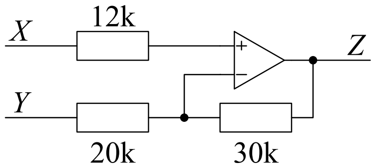


Figure 1.4

d) Assuming the opamp in the circuit of Figure 1.4 is ideal, give an expression for Z in terms of X and Y. [4]

[U] There is no current flowing through the $12\,\text{k}\Omega$ resistor, so $V_+ = X$. The circuit has negative feedback and so we also have $V_- = X$. Now, doing KCL at V_- gives

$$\frac{X-Y}{20} + \frac{X-Z}{30} = 0$$
$$\Rightarrow \quad 5X - 3Y - 2Z = 0$$
$$\Rightarrow \quad Z = 2.5X - 1.5Y$$

"An expression for Z" means an equation of the form "$Z = \ldots$"; the answer $2Z = 5X - 3Y$ is not what was requested so it only received full marks because the examiner was feeling generous. Quite a common mistake was to say $V_- = (Z-Y)\frac{20}{20+30}$; this expression uses the potential divider formula to calculate the voltage across the $20\,\text{k}\Omega$ resistor. However, since the leftmost end of the resistor is connected to Y rather than to 0, the correct expresion is $V_- = Y + (Z-Y)\frac{20}{20+30}$.

---

e) The diode in the circuit of Figure 1.5 has a forward voltage of $0.7\,\text{V}$ when conducting but is otherwise ideal. Determine the output voltage, Y, when
(i) $X = 1\,\text{V}$,
(ii) $X = 5\,\text{V}$
(iii) $X = -5\,\text{V}$. [5]

[U] If the diode is not conducting, then the circuit is a potential divider and $Y = 0.75X$ and the voltage across the diode is $0.25X$. Thus, the diode will be off when $0.25X < 0.7 \Rightarrow X < 2.8\,\text{V}$. If the diode is conducting, then $Y = X - 0.7$.

(i) when $X = 1\,\text{V}$, the diode is off and $Y = 0.75X = 0.75\,\text{V}$. (ii) when $X = 5\,\text{V}$, the diode is conducting and $Y = X - 0.7 = 4.3\,\text{V}$. (iii) when $X = -5\,\text{V}$, the diode is off and $Y = 0.75X = -3.75\,\text{V}$.

Several people thought the diode was conducting in part (i) and gave the incorrect answer of $Y = 0.3$. In fact, the diode is not conducting and its voltage is $+0.25$ which, although positive, is less than the 0.7 required to make the diode conduct (as stated in the question). Quite a few people inverted the sign of the diode voltage to give $Y = X + 0.7$; this would mean that the diode supplies energy when a current flows through it in the direction of the arrow.

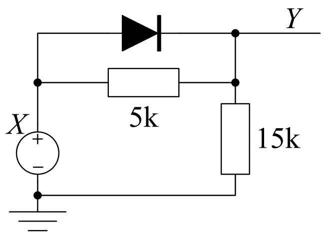


Figure 1.5

f) i) The diagram of Figure 1.6 shows an AC source with r.m.s. voltage 230 V driving a load with impedance $50+25j\,\Omega$ through a line with impedance $2\,\Omega$.

Determine the complex power, given by $S = \widetilde{V} \times \widetilde{I}^*$, absorbed by the load and the complex power absorbed by the 2Ω resistor. [4]

[U] The current phasor is $\widetilde{I} = \frac{\widetilde{V}}{52+25j} = 3.593 - 1.727j$. *The complex power absorbed by an impedance is* $S = \widetilde{V} \times \widetilde{I}^* = \left|\widetilde{I}\right|^2 Z = 15.891Z$. *So the power absorbed by the resistor is* $S_R = 15.891 \times 2 = 31.781\,\mathrm{W}$. *The power absorbed by the load is* $S_L = 15.891 \times (50+25j) = 794.5+397.3j\,\mathrm{VA}$.

Learning how to use the complex arithmetic functions in the calculator is recommended; it saves a lot of time and reduces arithmetic errors. The complex power absorbed by an impedance, Z, is $S = \widetilde{V} \times \widetilde{I}^* = \left|\widetilde{I}\right|^2 Z = \frac{|\widetilde{V}|^2}{Z^*}$*; however it is essential to use the correct* $\widetilde{V}$ *and/or* $\widetilde{I}$. *Some people used* $\frac{|\widetilde{V}|^2}{Z^*}$ *with* $\widetilde{V} = 230$ *but this is incorrect because* 230 *is not the voltage across either the load or the resistor. Quite a lot of people used an incorrect formula for S such as:* $\frac{|\widetilde{V}|^2}{Z}$, $\widetilde{I}^2 Z$, $\frac{\widetilde{I}^2}{Z}$, $\frac{\widetilde{V}}{Z^*}^2$ *or* $\frac{\widetilde{V}}{Z}^2$. *Several of these incorrect expressions wrongly give a complex value for S even when Z is real. The complex power absorbed by a resistor is always real; indeed, for any impedance, the argument of S equals the argument of Z as is obvious from* $S = \left|\widetilde{I}\right|^2 Z$. *Rounding errors mant that several people gave answers like* $S_R = \widetilde{V} \times \widetilde{I}^* = 31.781 + 0.0003j$ *which was awarded full marks but is nevertheless a bit weird.*

---

ii) A capacitor with impedance $-200j$ is now connected across the load, as indicated in Figure 1.7. Determine the complex power absorbed by the load and the complex power absorbed by the 2Ω resistor. [4]

[U] The combined load+capacitor impedance is now $Z_{LC} = \frac{-200j(50+25j)}{50+25j-200j} = 60.38+11.32j\,\Omega$. *The source current is now* $\tilde{I}_R = \frac{V}{2+Z_{LC}} = \frac{230}{62.38+11.32j} = 3.570 - 0.648j$ *which means that the voltage across the resistor is* $\tilde{V}_R = (3.570-0.648j) \times 2 = 7.1393 - 1.2957j$. *So the voltage across the load+capacitor is* $\tilde{V}_{LC} = \widetilde{V} - \tilde{V}_R = 222.86 + 1.2957j$ *or, via another route,* $\tilde{V}_{LC} = \frac{Z_{LC}}{2+Z_{LC}} \times \widetilde{V} = \frac{(60.38+11.32j)230}{62.38+11.32j} = 222.86+1.2957j$.

So the power absorbed by the resistor is $S_R = \left|\tilde{I}_R\right|^2 \times R = |3.570-0.648j|^2 \times 2 = 13.162 \times 2 = 26.32\,\mathrm{W}$, *a decrease of* 17%. *The power absorbed by the load is* $S_L = \frac{|\tilde{V}_L|^2}{Z_L^*} = \frac{|222.86+129.57j|^2}{50-25j} = \frac{49669}{50-25j} = 794.7+397.3j\,\mathrm{VA}$ *which is almost exactly the same as before.*

Generally people found this much harder than the previous part and quite a lot of people did not even attempt the question. Some took the capacitor impedance as $\frac{1}{-200j}$ *instead of* $200j$ *(presumably thinking* *of $\frac{1}{j\omega C}$).This gave a combined impedance of around $0.005j$ which is so small that it should have rung alarm bells. Many people calculated the source current correctly but then used $\widetilde{I}^2Z$ to calculate the complex power absorbed by the load forgetting that not all the current flows throught he load. Surprisingly many people calculated the power assuming that $\widetilde{V} = 230$ was the voltage across the resistor and/or the load. Several people thought that, because a capacitor does not absorb and energy on average, the overall power consumption of the circuit would remain unchanged and/or the load voltage would remain unchanged: this is not true because the currents through the other components change in both phase and magnitude. Some people left their answers as complicated arithmetic expressions rather than as decimal numbers despite the "Instructions for Candidates" at the start of the paper.*

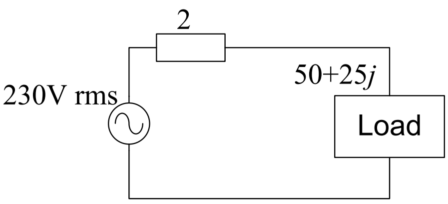


Figure 1.6

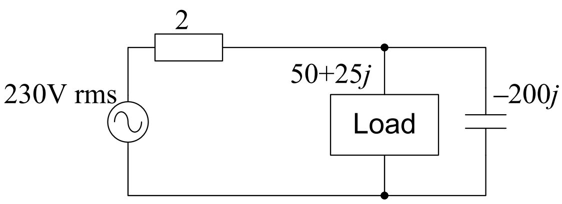


Figure 1.7

g) Determine the gain, $\frac{Y}{X}$, for the block diagram shown in Figure 1.8. The rectangular blocks are drawn with inputs at the left and outputs at the right and have gains of F, G and H respectively. The open circle represents an adder/subtractor; its three inputs have the signs indicated on the diagram and its output is V. [4]

[U] We can write down the following equations from the block diagram:

$$\begin{aligned} V &= X - Y - FHV \\ Y &= FGV \end{aligned}$$

We need to eliminate V from these equations:

$$\begin{aligned} V &= \frac{Y}{FG} \\ \Rightarrow \frac{1}{FG}Y &= X - Y - \frac{H}{G}Y \\ \Rightarrow \left(\frac{1}{FG} + 1 + \frac{H}{G}\right)Y &= X \\ \frac{1+FG+FH}{FG}Y &= X \\ \frac{Y}{X} &= \frac{FG}{1+F(G+H)} \end{aligned}$$

An even easier derivation, if you spot it is to write

$$V = X - FGV - FHV$$
$$\Rightarrow X = V + FGV + FHV$$
$$\Rightarrow \frac{Y}{X} = \frac{FGV}{V + FGV + FHV} = \frac{FG}{1 + FG + FH}$$

Some people did not have a clear goal when doing their algebra and ended up with an answer that included V. You start with two equations and two unknowns (V and Y) and the entire purpose of the algebra is to eliminate V. Soe gave an additional label, W, to the node joining F and G; this is not wrong but it results in three simultaneous equations instead of only two.

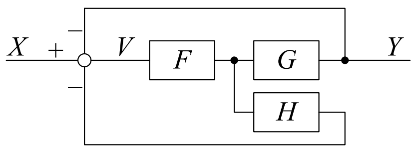


Figure 1.8

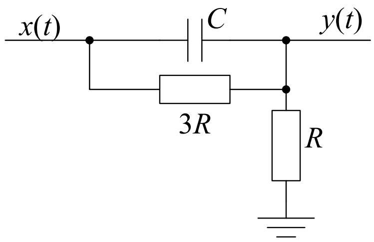


Figure 1.9

h) The input voltage in Figure 1.9 is given by

$$x(t) = \begin{cases} 0 & t < 0 \\ 8\,\mathrm{V} & t \geq 0. \end{cases}$$

i) Determine the time constant of the circuit. [2]

[U] The time constant is given by $\tau = R_{Thev}C$ where R_{Thev} is the Thévenin resistance across the terminals of the capacitor. If we short circuit the source, $x(t)$, we find $R_{Thev} = \frac{3R \times R}{3R + R} = 0.75R$ so the time constant is $\tau = 0.75RC$.

An alternative method is to calculate the transfer function of the circuit as

$$\frac{Y}{X} = \frac{R}{R + \frac{1}{j\omega C + \frac{1}{3R}}} = \frac{R}{R + \frac{3R}{j\omega 3RC + 1}} = \frac{j\omega 3RC + 1}{j\omega 3RC + 1 + 3} = \frac{j\omega 3RC + 1}{j\omega 3RC + 4}$$

from which the time constant is the reciprocal of the denominator corner frequency and therefore equals $\tau = 0.75RC$. The transfer function also gives the DC gain as 0.25 and the HF gain as 1.

Most got this right. The transfer function method involves quite a bit more effort for this circuit and sometimes led to algebra errors.

---

ii) Determine an expression for $y(t)$ for $t > 0$. [5]

[U] Since the DC gain of the circuit is 0.25 (obtained by treating the capacitor as an open circuit), the steady state output for $t \geq 0$ *is* $y_{SS}(t) = 0.25x(t) = 2$.

At time $t = 0$, *the capacitor voltage,* $y - x$, *cannot change instantaneously. Therefore,* $y(0+) - x(0+) = y(0-) - x(0-) = 0$ *and hence* $y(0+) = x(0+) = 8$. *The transient amplitude is therefore* $y(0+) - y_{SS}(0+) = 8 - 2 = 6$. *The complete output is therefore* $y(t) = 2 + 6e^{-\frac{t}{\tau}}$.

Alternatively, the transfer function is $H(j\omega) = \frac{1+3j\omega RC}{4+3j\omega RC}$ *from which* $H(\infty) = 1$. *Hence* $\Delta y = 1 \times \Delta x = 8$ *and so* $y(0+) = y(0-) + 8 = 8$.

Quite a few people assumed that capacitor voltage continuity implied $y(0+) = y(0-) = 0$ *which is not true. Apart from this, most people got this question right.*

---

2. The frequency response of a circuit is given by

$$H(j\omega) = \frac{aj\omega}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2}$$

where a, ζ and ω_0 are real numbers.

a) i) By dividing the numerator and denominator of $H(j\omega)$ by $j\omega$ and then multiplying the resultant expression by its complex conjugate, show that $|H(j\omega)|^2 = \dfrac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2}$. [3]

[U] Dividing numerator and denominator by $j\omega$ gives

$$H(j\omega) = \frac{a}{2\zeta\omega_0 + j\omega + \frac{\omega_0^2}{j\omega}} = \frac{a}{2\zeta\omega_0 + j\left(\omega - \frac{\omega_0^2}{\omega}\right)}.$$

To multiply by its complex conjugate we take the sum of the real and imaginary parts in both numerator and denominator to obtain

$$|H(j\omega)|^2 = \frac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2}.$$

Mostly OK although quite a few people seemed unfamiliar with the very useful property of complex numbers that if $z = a + jb$, then $zz^* = |z|^2 = a^2 + b^2$. Several people rationalized the fraction in the second line above by multiplying numerator and denominator by $2\zeta\omega_0 - j\left(\omega - \frac{\omega_0^2}{\omega}\right)$; although this is mathematically valid it is usually a bad thing to do algebraically because it doubles the polynomial order of the denominator. In this case, it makes the algebra much harder. Several people didn't like drawing ζ; one person just used τ instead.

---

ii) Explain why the maximum value of $|H(j\omega)|^2$ occurs when the quantity $\left(\omega - \frac{\omega_0^2}{\omega}\right)$ equals zero. Hence show that the maximum occurs at $\omega = \omega_0$ and determine $|H(j\omega_0)|^2$. [2]

[U] The denominator of $|H(j\omega)|^2$ is the sum of two squares which are both always ≥ 0. Only one of the squares involves ω while the other is constant. Therefore the denominator in minimized (and $|H(j\omega)|^2$ is maximized) when this term is zero:

$$\left(\omega - \frac{\omega_0^2}{\omega}\right)^2 = 0 \quad\Rightarrow\quad \omega = \frac{\omega_0^2}{\omega} \quad\Rightarrow\quad \omega = \pm\omega_0.$$

Substituting this into the expression for $|H(j\omega)|^2$ gives

$$\max\left\{|H(j\omega)|^2\right\} = \frac{a^2}{4\zeta^2\omega_0^2}.$$

A valid but somewhat more laborious method is to differentiate the expression for $|H(j\omega)|^{-2}$ or, even more laboriously, $|H(j\omega)|^2$ and find when it is zero. The entire purpose of the manipulations in part (i) was to avoid the need for this by pushing all dependency on ω into a single non-negative term.

Most got this right although quite a few people gave non-algebraic reasons, which are not true for all circuits, such as "The peak of $|H(j\omega)|^2$ must be at the corner frequency, ω_0" rather than giving an algebrai explanation. Very few people commented on the importance of the first denominator term, $4\zeta^2\omega_0^2$, being positive. Quite a lot of people lost a mark becuase they did not give the value of $\max\left\{|H(j\omega)|^2\right\}$ as requested.

---

iii) Find expressions for the two positive values of ω for which $|H(j\omega)|^2 = \frac{a^2}{8\zeta^2\omega_0^2}$ and determine a simplified expression for the difference between them. [4]

[U] We have

$$
\begin{aligned}
|H(j\omega)|^2 = \frac{a^2}{4\zeta^2\omega_0^2 + \left(\omega - \frac{\omega_0^2}{\omega}\right)^2} &= \frac{a^2}{8\zeta^2\omega_0^2} \\
\Rightarrow \quad \left(\omega - \frac{\omega_0^2}{\omega}\right)^2 &= 4\zeta^2\omega_0^2 \\
\omega - \frac{\omega_0^2}{\omega} &= \pm 2\zeta\omega_0 \\
\omega^2 \pm 2\zeta\omega_0\omega - \omega_0^2 &= 0 \\
\omega &= \frac{\pm 2\zeta\omega_0 \pm \sqrt{4\zeta^2\omega_0^2 + 4\omega_0^2}}{2} \\
&= \pm\zeta\omega_0 \pm \sqrt{\zeta^2\omega_0^2 + \omega_0^2} \\
&= \left(\pm\zeta \pm \sqrt{\zeta^2+1}\right)\omega_0.
\end{aligned}
$$

This gives a total of four roots: two positive and two negative. Since the square-root term is larger in magnitude than the first term, the two positive roots will be when the square root term is positive:

$$
\omega_{1,2} = \pm\zeta\omega_0 + \sqrt{\zeta^2\omega_0^2 + \omega_0^2} = \left(\pm\zeta + \sqrt{\zeta^2+1}\right)\omega_0
$$

Thus the difference between these two roots will be $\omega_2 - \omega_1 = 2\zeta\omega_0$ (since the square root term cancels out in the subtraction). At these values of ω, the response has fallen 3 dB from its peak, so this difference is the 3 dB bandwidth.

If you don't take the square root in the third line above, you get a *quadratic in ω^2 which is rather messier to solve (especially if you don't know in advance the correct answer to aim for):*

$$\begin{aligned}
\left(\omega - \frac{\omega_0^2}{\omega}\right)^2 &= 4\zeta^2\omega_0^2 \\
\omega^2 - 2\omega_0^2 + \frac{\omega_0^4}{\omega^2} - 4\zeta^2\omega_0^2 &= 0 \\
\omega^4 - 2\omega_0^2\left(1+2\zeta^2\right)\omega^2 + \omega_0^4 &= 0 \\
\Rightarrow \quad \omega^2 &= \frac{2\omega_0^2\left(1+2\zeta^2\right) \pm \sqrt{4\omega_0^4\left(1+2\zeta^2\right)^2 - 4\omega_0^4}}{2} \\
&= \omega_0^2\left(1+2\zeta^2\right) \pm \omega_0^2\sqrt{\left(1+2\zeta^2\right)^2 - 1} \\
&= \left(1+2\zeta^2 \pm 2\zeta\sqrt{\zeta^2+1}\right)\omega_0^2 \\
&= \left(\zeta \pm \sqrt{\zeta^2+1}\right)^2\omega_0^2 \\
\Rightarrow \quad \omega &= \pm\left(\zeta \pm \sqrt{\zeta^2+1}\right)\omega_0
\end{aligned}$$

Very many people omitted the $\pm$in the third line: $\omega - \frac{\omega_0^2}{\omega} = \pm 2\zeta\omega_0$ and therefore did not find both of the positive solutions; whenever you take a square root in algebra, you need to insert a $\pm$ sign. Others successfully found all four solutions but took the wrong pair of roots from $\omega = \left(\pm\zeta \pm \sqrt{\zeta^2+1}\right)\omega_0$ by setting the first $\pm$ to be always positive. Quite a few people got a sign wrong and wrote $\sqrt{4\zeta^2\omega_0^2 - 4\omega_0^2}$ instead of $\sqrt{4\zeta^2\omega_0^2 + 4\omega_0^2}$ when writing the solution to the quadratic equation.

---

b) Suppose now that $a = 5000\,\mathrm{s}^{-1}$, $\zeta = 0.1$ and $\omega_0 = 5000\,\mathrm{rad/s}$.

i) Determine the low and high frequency asymptotes of $H(j\omega)$. [2]

[U] The LF asymptote is found by taking the terms with the lowest power of $j\omega$ in numerator and denominator and is

$$H_\mathrm{L}(j\omega) = ja\omega_0^{-2}\omega = \frac{j\omega}{5000} = j2\times 10^{-4}\omega$$

. Similarly, the HF asymptote is

$$H_\mathrm{H}(j\omega) = -ja\omega^{-1} = \frac{5000}{j\omega} = -j5000\omega^{-1}$$

.

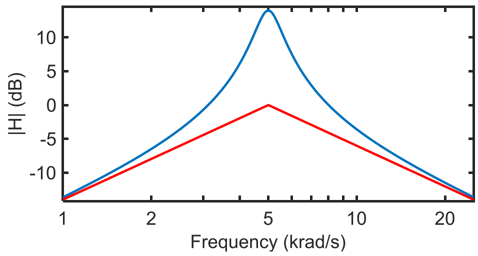
Mostly OK although some people omitted the "j" factors even though the question asked for the asymptotes of $H(j\omega)$ rather than of $|H(j\omega)|$. A few people said the LF asymptote was $H_\mathrm{L}(j\omega) = 0$ because there is no constant term in the numerator. A few people noticed that the units of a were in s^{-1} and wrongly multiplied by 2πin order "to convert to $\mathrm{rad/s}$". A few people gave the asymptotes as $H_\mathrm{L}(j\omega) =$

$H_H(j\omega) = 0$; note that the asymptote, $H_L(j\omega)$ is a tangent to the curve of $H(j\omega)$ at low frequencies and it is not the same thing as $H(j0)$ which is just a single value of $H(j\omega)$.

---

ii) Draw a dimensioned sketch showing the high and low frequency asymptotes as well as the true magnitude response, $|H(j\omega)|$. Indicate on your graph in dB the peak value of $|H(j\omega)|$ and the value of the asymptotes at their point of intersection. [5]

[U] The magnitude asymptotes cross when $\left|ja\omega_0^{-2}\omega\right| = \left|-ja\omega^{-1}\right| \Rightarrow a\omega_0^{-2}\omega = a\omega^{-1} \Rightarrow \omega = \omega_0 = 5000$. At this point, their value is $a\omega_0^{-1} = 1 = 0\,\text{dB}$. From part ii), the peak magnitude gain is $\sqrt{\frac{a^2}{4\zeta^2\omega_0^2}} = \frac{a}{2\zeta\omega_0} = 5 = 14\,\text{dB}$ at $\omega = 5000$. We also know from part iii) that the 3 dB bandwidth is $2\zeta\omega_0 = 1000\,\text{rad/s}$. Thus we can draw the graph as shown.

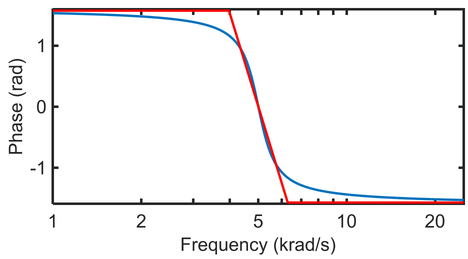


Surprisingly many people got the gain at the asymptote interestection wrong; often this was set equal to 5 or 14 dB and, in one case, 100 dB which is rather massive. Quite a few people had the true peak at 28 dB because part a-i asked for $|H(j\omega)|^2$ rather than $|H(j\omega)|$. Some people drew the "true" curve nowhere near the asymptotes; the main property of asymptotes is that the true curve tends to the asymptote at frequencies far from the corner frequency. Quite a few people drew the response of a low pass filter (i.e. with a low frequency asymptote of 1) evennthough they correctly calculated the asymptotes in the previous part.

---

iii) Draw a dimensioned sketch of the straight-line approximation to the phase response, $\angle H(j\omega)$. You may assume without proof that the gradient of the approximation at ω_0 is equal to $-0.5\pi\zeta^{-1}$ radians per decade where "decade" means a factor of 10 in frequency. [4]

[U] From part i), the LF and HF phase shifts are $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$. Also, at $\omega = \omega_0$, the outer terms of the quadratic cancel and the phase shift is 0. At ω_0, the gradient is $-0.5\pi\zeta^{-1} = -15.71$ (meaning that it changes by π in 2ζ decades), so the central line of the approximation will hit $\pm\frac{\pi}{2}$ at $\omega = \omega_0 \pm \zeta$ decades. $\zeta = 0.1$ decades is equivalent to a multiple of $10^{0.1} = 1.259$. So the sloping segement goes between $\omega = [5000 \div 1.29, 5000 \times 1.29] = [3972, 6295]$.



Many people omitted this part entirely but most f those who attempted it got the overal shape of the graph right. Quite a few people showed the phase response with completely the wrong shape and with LF and HF asymptotes having the same slopes as the magnitude response. The LF and HF phases can be deduced from the answers to part b-i: $\frac{j\omega}{5000}$ has a phase of $+\frac{\pi}{2}$ while $\frac{5000}{j\omega}$ has a phase of $-\frac{\pi}{2}$. Many people had gradient changes at $\omega = \{500, 50\text{k}\}$ which gives an intermediate slope of -0.5π; this would be the slope if $\zeta = 1$.The cornbers in the phase response are at $\omega = \omega_0 \pm \zeta$ decades.

---

c) i) Show that the frequency response, $\frac{Y(j\omega)}{X(j\omega)}$ of the circuit shown in Figure 2.1 is given by [5]

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega R_2 C}{(j\omega)^2 R_1 R_2 C^2 + 2j\omega R_1 C + 1}.$$

[U] KCL at the –ve opamp input (which is a virtual ground) gives

$$j\omega C(0-V) + \frac{0-Y}{R_2} = 0$$
$$\Rightarrow \quad V = \frac{-Y}{j\omega R_2 C}.$$

This can also be viewed as an inverting opamp circuit with V as the input signal.

KCL at V gives

$$\frac{V-X}{R_1} + j\omega C(V-Y) + j\omega C(V-0) = 0$$
$$V(1+2j\omega R_1 C) - X - j\omega R_1 C Y = 0.$$

Substituting from the first equation gives

$$\frac{-Y}{j\omega R_2 C}(1+2j\omega R_1 C) - X - j\omega R_1 C Y = 0.$$
$$\left(1 + 2j\omega R_1 C + (j\omega)^2 R_1 R_2 C^2\right) Y = -j\omega R_2 C X$$
$$\Rightarrow \quad \frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega R_2 C}{(j\omega)^2 R_1 R_2 C^2 + 2j\omega R_1 C + 1}.$$

A neater, but slightly trixier, soluution method is

$$
\begin{aligned}
Y &= -j\omega R_2 C V\\
X &= (1+2j\omega R_1 C)V - j\omega R_1 C Y\\
&= (1+2j\omega R_1 C)V + (j\omega)^2 R_1 R_2 C^2 V\\
\frac{Y}{X} &= \frac{-j\omega R_2 C V}{(1+2j\omega R_1 C)V + (j\omega)^2 R_1 R_2 C^2 V}\\
&= \frac{-j\omega R_2 C}{(j\omega)^2 R_1 R_2 C^2 + 2j\omega R_1 C + 1}.
\end{aligned}
$$

Most got this right although a few people needed to fudge the minus sign in the numerator. Several people did KCL at node Y which doesn't work because the opamp output current is unknown (and was ignored by those who did this). A few people wrote down the correct initial equations but did not have a clear goal for their algebra (namely to eliminate V between the equations) and often ended you with a formula that sill included V. One or two people tried to use the formula for an inverting opamp with some combination of C, C and R_2 as the feedback network; this is not valid because node V is not the opamp input terminal. Several people wrote the second KCL equation as $\frac{V-X}{R_1} + j\omega C(V-Y) + \frac{j\omega C}{1+j\omega R_2 C}(V-Y) = 0$ where the last term arisies from the series combination of C and R_2 (which is valid because no current is drawn by the V_- terminal); this equation is valid but it complicates the algebra.

---

ii) Determine simplified expressions for a, ζ and ω_0 so that the expression given in part c)i) equals that given above for $H(j\omega)$. [3]

[U] In order to make the coefficient of $(j\omega)^2$ equal to unity (to match the equation for $H(j\omega)$), we divide numerator and denominator by $R_1 R_2 C^2$ to obtain

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{-(R_1 C)^{-1} j\omega}{(j\omega)^2 + 2(R_2 C)^{-1} j\omega + (R_1 R_2 C^2)^{-1}}.$$

Matching coefficients gives

$$
\begin{aligned}
a &= -(R_1 C)^{-1}\\
\omega_0 &= \left(R_1 R_2 C^2\right)^{-0.5} = \frac{1}{C\sqrt{R_1 R_2}}\\
\zeta &= \frac{1}{R_2 C \omega_0} = R_1 C \omega_0 = \sqrt{\frac{R_1}{R_2}}.
\end{aligned}
$$

Surprisingly many people didn't realize that they needed to divide numerator and denominator by $R_1 R_2 C^2$ to make the coefficient of $(j\omega)^2$ equal to unity. Identifyng the remaining coefficients then resulted in a fixed corner frequency of $\omega_0 = 1$ which is constradicted by part c-iii and is very unlikely in any case. Several people got mixed *up between "a" in the question and "a" in the standard quadratic expression ax^2+bx+c; they are completely different. Quite a lot of people wrote $\zeta = \frac{\sqrt{R_1R_2}}{R_2}$ or even $\frac{\sqrt{R_1R_2C^2}}{R_2C}$ rather than the simpler expression $\sqrt{\frac{R_1}{R_2}}$ which surprised me since the question asked for "simplified expressions".*

---

iii) Given that $C = 10\,\text{nF}$, determine the values of R_1 and R_2 so that $\omega_0 = 5000\,\text{rad/s}$ and $\zeta = 0.1$. [2]

[U] From part ii),

$$\zeta = \frac{1}{R_2C\omega_0} \quad \Rightarrow \quad R_2 = \frac{1}{\zeta C\omega_0} = 200\,\text{k}\Omega.$$

Now we can write

$$\zeta = \sqrt{\frac{R_1}{R_2}} \quad \Rightarrow \quad R_1 = R_2\zeta^2 = \frac{\zeta}{C\omega_0} = 0.012 = 2\,\text{k}\Omega.$$

Most people did this part OK although there were quite often algebra errors. There typically resulted in absurd values for the resistors like $10^{10}\,\Omega$ or $10^{-5}\,\Omega$ which is a pretty sure sign of an error. One common mistake was in calculating $\omega_0C = 5\times10^{-5}$; many people got the exponent wrong.

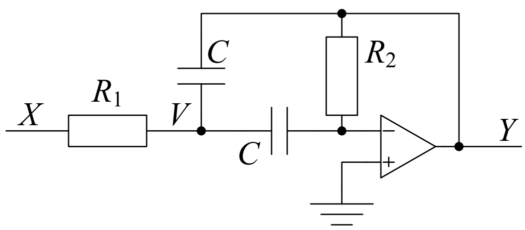


Figure 2.1

3. Figure 3.1 shows a shows a transmission line of length $L = 10\,\text{m}$ whose characteristic impedance is $Z_0 = 120\,\Omega$ and whose propagation velocity is $u = 2 \times 10^8\,\text{m/s}$. Distance along the line is denoted by x and the two points $x = 0$ and $x = L$ are marked in the figure.

At a point x on the line, the line voltage and current are given by $v_x(t) = f_x(t) + g_x(t)$ and $i_x(t) = Z_0^{-1}(f_x(t) - g_x(t))$ where $f_x(t) = f_0(t - u^{-1}x)$ and $g_x(t) = g_0(t + u^{-1}x)$ are the forward and backward waves respectively.

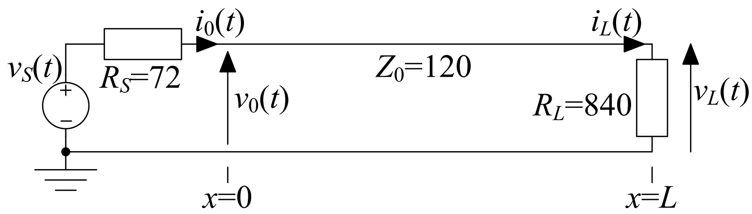


Figure 3.1

a) i) At the position $x = L$, the backward wave is given by $g_L(t) = \rho_L f_L(t)$ where $\rho_L = 0.75$ is the reflection coefficient at $x = L$.

Show that $g_0(t) = \rho_L f_0(t - 2u^{-1}L)$. [3]

[B] We substitute the given expressions, $f_x(t) = f_0(t - u^{-1}x)$ and $g_x(t) = g_0(t + u^{-1}x)$ into $g_L(t) = \rho_L f_L(t)$ to obtain

$$\begin{aligned} g_L(t) &= \rho_L f_L(t) \\ g_0(t + u^{-1}L) &= \rho_L f_0(t - u^{-1}L) \\ g_0(t') &= \rho_L f_0(t' - 2u^{-1}L) \end{aligned}$$

where in the final line we make the substitution $t' = t + u^{-1}L$.

Most people got the general idea right. Many people explained the reason in words rather than proving it algebraically. Rather few people formally made the substitution $t' = t + u^{-1}L$ or its equivalent which is requred for going rigorously from line 2 to line 3 above. Some people converted to phasor form in order to prove the result so that the second line became $G_0 e^{jkL} = \rho_L F_0 e^{-jkL} \Rightarrow G_0 = \rho_L F_0 e^{-j2kL}$. The problem with this approach is that while the formula in the question applies to any sort of waveform, phasors can only be used if $f_0(t)$ is a sine wave.

---

ii) At $x = 0$, show that $v_s(t) = v_0(t) + R_S i_0(t)$. Hence show that $f_0(t)$ can be written in the form $f_0(t) = \tau_0 v_s(t) + \rho_0 g_0(t)$ and determine the numerical values of τ_0 and ρ_0. [6]

[U] Applying Kirchoff's Current law at the rightmost end of R_S gives $\frac{v_0 - v_s}{R_S} + i_0 = 0$ from which $v_s = v_0 + R_S i_0$.

Substituting for v_0 and i_0 (using the formulae given in the preamble) *results in*

$$\begin{aligned} v_s(t) &= (f_0(t) + g_0(t)) + R_S Z_0^{-1}(f_0(t) - g_0(t)) \\ &= \left(1 + R_S Z_0^{-1}\right) f_0(t) + \left(1 - R_S Z_0^{-1}\right) g_0(t) \\ \Rightarrow \quad f_0(t) &= \frac{1}{1 + R_S Z_0^{-1}} v_s(t) - \frac{1 - R_S Z_0^{-1}}{1 + R_S Z_0^{-1}} g_0(t) \\ &= \frac{Z_0}{R_S + Z_0} v_s(t) + \frac{R_S - Z_0}{R_S + Z_0} g_0(t) \end{aligned}$$

from which $\tau_0 = \frac{Z_0}{R_S+Z_0} = \frac{120}{192} = 0.625$ *and* $\rho_0 = \frac{R_S-Z_0}{R_S+Z_0} = \frac{-48}{192} = -0.25$.

Several people omitted the minus sign from the numerical value of ρ_0 *and several more omitted the numerical values entirely. Several people decomposed* v_S *as* $v_S(t) = f_S(t) + g_S(t)$ *but the forward and backward waves exist only on the transmission line and* v_S *is not part of the line (it is separated from the line by* R_S*) so* f_S *and* g_S *have no meaning. Proving the first part,* $v_s = v_0 + R_S i_0$*, is very easy but some people lost easy marks because they did not do it.*

---

iii) By combining the results of parts i) and ii) show that

$$f_0(t) = \tau_0 v_s(t) + \rho_0 \rho_L f_0(t - 2u^{-1}L).$$

Hence prove, by using induction or otherwise, that

$$f_0(t) = \sum_{n=0}^{\infty} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right).$$ [6]

[U] Substituting part i) into part ii) gives $f_0(t) = \tau_0 v_s(t) + \rho_0 \rho_L f_0(t - 2u^{-1}L)$ *directly.*

The informal way of proving the result is to use the above equation to substitute repeatedly for the $f_0(\ldots)$ *factor in the final term of the equation itself:*

$$\begin{aligned} f_0(t) &= \tau_0 v_s(t) + \rho_0 \rho_L f_0(t - 2u^{-1}L) \\ &= \tau_0 v_s(t) + \rho_0 \rho_L \tau_0 v_s(t - 2u^{-1}L) + \rho_0^2 \rho_L^2 f_0(t - 4u^{-1}L) \\ &= \underbrace{\tau_0 v_s(t) + \rho_0 \rho_L \tau_0 v_s(t - 2u^{-1}L) + \rho_0^2 \rho_L^2 \tau_0 v_s(t - 4u^{-1}L)}_{\sum_{n=0}^{2} \tau_0 \rho_0^n \rho_L^n v_s(t - 2nu^{-1}L)} + \rho_0^3 \rho_L^3 f_0(t - 6u^{-1}L) \\ &= \ldots \text{ and so on } \ldots \\ &= \sum_{n=0}^{\infty} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right). \end{aligned}$$

Mathematical induction is the formal way of making this argument rigorous. We prove by induction that, for any $N \geq 1$*,the following proposition is true:*

$$f_0(t) = \left(\sum_{n=0}^{N-1} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right) \right) + \rho_0^N \rho_L^N f_0\left(t - 2Nu^{-1}L\right).$$

This corresponds to line N above with the first N terms combined into a summation.

When $N=1$, this is true because the summation has only one term and it becomes the result given in the question.

We now assume it is true for $N=N_0$ and prove it for $N=N_0+1$ by substituting the result from the first line into the final term:

$$\begin{aligned}
f_0(t) &= \left(\sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right)\right) + \rho_0^{N_0}\rho_L^{N_0}\left\{f_0\left(t-2N_0u^{-1}L\right)\right\} \\
&= \left(\sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right)\right) \\
&\quad + \rho_0^{N_0}\rho_L^{N_0}\left\{\tau_0 v_s(t-2N_0u^{-1}L) + \rho_0\rho_L f_0(t-2N_0u^{-1}L-2u^{-1}L)\right\} \\
&= \left(\sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right)\right) \\
&\quad + \tau_0\rho_0^{N_0}\rho_L^{N_0} v_s(t-2N_0u^{-1}L) + \rho_0^{N_0+1}\rho_L^{N_0+1} f_0(t-2N_0u^{-1}L-2u^{-1}L) \\
&= \left(\sum_{n=0}^{N_0} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right)\right) + \rho_0^{N_0+1}\rho_L^{N_0+1} f_0(t-2\left(N_0+1\right)u^{-1}L)
\end{aligned}$$

As $N_0\to\infty$, the final term tends to zero because $|\rho_0|, |\rho_L| < 1$ from which

$$f_0(t) = \sum_{n=0}^{\infty} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right).$$

An alternative, less general, approach is to assume that $f_0(t) = v_s(t) = 0$ for t<0. Then the induction proposition for $N=N_0$ can be that

$$f_0(t) = \sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right) \quad \text{for} \quad t < 2N_0u^{-1}L$$

which is a simpler formula than that of the previous proposition albeit with a time constraint. The case for $N_0=1$ is now $f_0(t) = \tau_0 v_s(t)$ for $t<2u^{-1}L$; this is true because the second term in the original formula is always zero for this restriction on t. Now we can write, for $t<2\left(N_0+1\right)u^{-1}L$,

$$\begin{aligned}
f_0(t) &= \tau_0 v_s(t) + \rho_0\rho_L f_0(t-2u^{-1}L) \\
&= \tau_0 v_s(t) + \rho_0\rho_L \sum_{n=0}^{N_0-1} \tau_0\rho_0^n\rho_L^n v_s\left((t-2u^{-1}L)-2nu^{-1}L\right) \\
&= \tau_0 v_s(t) + \sum_{n=0}^{N_0-1} \tau_0\rho_0^{n+1}\rho_L^{n+1} v_s\left(t-2\left(n+1\right)u^{-1}L\right) \\
&= \tau_0 v_s(t) + \sum_{n'=1}^{N_0} \tau_0\rho_0^{n'}\rho_L^{n'} v_s\left(t-2n'u^{-1}L\right) \\
&= \sum_{n'=0}^{N_0} \tau_0\rho_0^{n'}\rho_L^{n'} v_s\left(t-2n'u^{-1}L\right)
\end{aligned}$$

where the second line is valid because if $t<2\left(N_0+1\right)u^{-1}L$ then $(t-2u^{-1}L) < 2N_0u^{-1}L$ as required by the induction proposition for $N=N_0$. When $N=\infty$, we get $f_0(t) = \sum_{n=0}^{\infty} \tau_0\rho_0^n\rho_L^n v_s\left(t-2nu^{-1}L\right)$ for $t<\infty$.

Most people tried to do this using induction but very few gave a rigorous proof. The essential first step of an induction proof is to formulate a proposition that is true for all N; most people had difficulty in doing this. Many people started off with a statement like "$f_0(t) = \tau_0 v_S(t)$ is true" when it plainly isn't true since it contradicts the equation given in the question. Many people thought that $\sum_{n=0}^{\infty} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right)$ was a function of n which it isn't because n is just a dummy summation variable; so it doesn't make any sense to "assume this formula is true for $n = 0$". As discussed above, the formula $f_0(t) = \sum_{n=0}^{N_0-1} \tau_0 \rho_0^n \rho_L^n v_s\left(t - 2nu^{-1}L\right)$ is only true for $t < 2N_0 u^{-1}L$ and even then only if $f_0(t) = v_s(t) = 0$ for t<0. Many people assumed this formula was true for N_0 (without any extra conditions) and then "proved" that it was true for $N_0 + 1$.

---

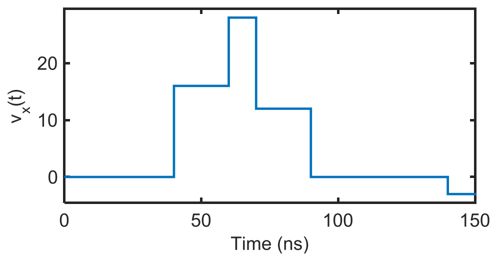
b) If the source is a 30 ns pulse given by

$$v_s(t) = \begin{cases} 25.6\,\text{V} & \text{for } 0 \le t \le 30\,\text{ns} \\ 0 & \text{otherwise} \end{cases},$$

draw a dimensioned sketch of the waveform $v_x(t)$ on the line at the point $x = 8\,\text{m}$ for the time interval $0 \le t \le 150\,\text{ns}$. Give the times of all discontinuities and the values of all horizontal portions of the waveform. [6]

[U] The propagatin velocity is $u = 2 \times 10^8$ which equals 5 ns per metre. So the pulse arrives at x at $8 \times 5 = 40\,\text{ns}$, reflects off the load and returns at $12 \times 5 = 60\,\text{ns}$. Subsequent arrivals are at these times pulse multiples of the round trip time, $20 \times 5 = 100\,\text{ns}$ so only the transition at $140\,\text{ns}$ lies within the plotted range. The initial forward wave amplitude is $25.6 \times \tau_0 = 16$ and subsequent amplitudes are $16 \times \rho_L = 12$, $12 \times \rho_0 = -3$, $-3 \times \rho_L = -2.25$.

putting all this together, we get transitions at $t = \{40, 60, 70, 90, 140\}$ of voltages $\delta v = \{16, 12, -16, -12, -3\}$. The voltage after each transition is therefore $v_x = \{16, 28, 12, 0, -3\}$.



Many people omitted the pulse that starts at $t = 140\,\text{ns}$; others got the amplitude wrong because they multiplied ρ_S by the amplitude of the original pulse rather than by that of the backward wave. Very many people omitted the multiplication by τ_0 which results in an amplitude of 25.6 for the forward wave (even though they had calculated τ_0 correctly in part a-ii). A few people used $\tau_0 = \frac{R_L}{R_S + R_L}$ instead of $\tau_0 = \frac{Z_0}{R_S + Z_0}$; until the wave first reaches the other end of the line, the value of R_L cannot affect anything.

---

c) Now assume that all voltages and currents are sinusoidal with angular frequency ω. The uppercase letter, V_x, denotes the phasor corresponding to $v_x(t)$.

i) The waveform $f_0(t) = A\cos(\omega t + \theta)$ is represented by the phasor $F_0 = Ae^{j\theta}$. Show that $F_x = F_0 e^{-jkx}$ where $k = u^{-1}\omega$. [3]

[B] We know that $f_x(t) = f_0(t - u^{-1}x) = 3A\cos\left(\omega(t - u^{-1}x) + \theta\right) = A\cos\left(\omega t + \theta - \omega u^{-1}x\right)$. The corresponding phasor is therefore $F_x = Ae^{j(\theta - \omega u^{-1}x)} = Ae^{j\theta}e^{-j\omega u^{-1}x} = F_0 e^{-jkx}$.

Mostly correct. Quite a few people got mixed up between the time-domain and the phasor-domain and wrote things like $f_x(t) = Ae^{j\left(\omega t + \theta - \omega u^{-1}x\right)}$ which falsely makes $f_x(t)$ complex-valued; when using phasors, you have made a mistake if you ever have "t" and "j" in the same equation. Also quite common was to write $f_x(t) = A\cos(\omega t + \theta - kx) = Ae^{j\theta}e^{-jkx}$ where the first two terms are real-valued waveforms but the last is a complex-valued phasor; fairly obviously this is not mathematically correct.

---

ii) By converting the first equation given in part a)iii) into phasor form, determine an expression for F_0 in terms of V_s. [3]

[U] Converting $f_0(t) = \tau_0 v_s(t) + \rho_0\rho_L f_0(t - 2u^{-1}L)$ into phasor form gives

$$F_0 = \tau_0 V_S + \rho_0\rho_L F_0 e^{-j2kL}$$
$$\Rightarrow \quad F_0\left(1 - \rho_0\rho_L e^{-j2kL}\right) = \tau_0 V_S$$
$$\Rightarrow \quad F_0 = \frac{\tau_0}{1 - \rho_0\rho_L e^{-j2kL}} V_S$$

For reasons that are unclear, several people wrote $e^{-2u^{-1}L}$ instead of $e^{-j2\omega u^{-1}L} = e^{-j2kL}$. Complex exponents are always dimensionless, so time is multiplied by ω (rad/s) and distance by k (rad/m). Several people gave an expression that involves an infinite sum: $F_0(t) = \sum_{n=0}^{\infty} \tau_0\rho_0^n\rho_L^n e^{-j2nkL}V_s$ which is correct and is the phasor form of the result of part a-iii. This is a geometric progression and can easily be shown to equal the simpler expression shown above in the solution. A big advantage of using phasors is that it eliminates such infinite sums. Surprisingly many people correctly wrote $F_0 = \tau_0 V_S + \rho_0\rho_L F_0 e^{-j2kL}$ but were unable to rearrange it into the form $F_0 = \ldots$ or else made algebraic errors when doing so.

---

iii) Determine an expression for V_x in terms of V_s. [3]

[U] We know that

$$
\begin{aligned}
V_x &= F_x + G_x \\
&= F_0 e^{-jkx} + G_0 e^{jkx} \\
&= F_0 e^{-jkx} + \rho_L F_0 e^{-j2kL} e^{jkx} \\
&= F_0 \left(e^{-jkx} + \rho_L e^{-jk(2L-x)} \right) \\
&= \frac{\tau_0 \left(e^{-jkx} + \rho_L e^{-jk(2L-x)} \right)}{1 - \rho_0 \rho_L e^{-j2kL}} V_S
\end{aligned}
$$

where the second line follows from $g_x(t) = g_0(t + u^{-1}x)$ *and the third from* $g_0(t) = \rho_L f_0(t - 2u^{-1}L)$.

Several people assumed $G_x = G_0 e^{-jkx}$ *instead of* $G_0 e^{+jkx}$. *Also many assumed that* $G_x = \rho_L F_x$ *but this is only true at* $x = L$; *the general relation is* $G_x = \rho_L F_x e^{-j2(L-x)}$ *which accounts for the round-trip distance to the end of the line and back.*