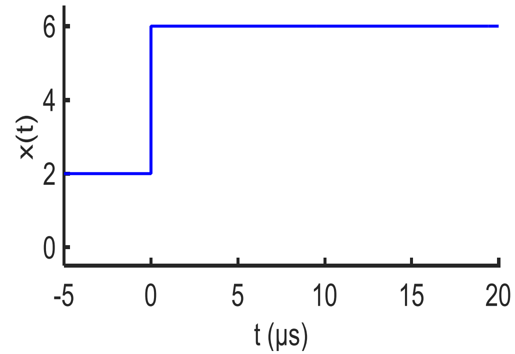
Revision Lecture 2: Transients & Lines

Transients: Basic Ideas

- Transients happen in response to a sudden change
  - Input voltage/current abruptly changes its magnitude, frequency or phase
  - A switch alters the circuit
- 1st order circuits only: one capacitor/inductor
- All voltage/current waveforms are: Steady State + Transient
  - Steady States: find with nodal analysis or transfer function
    - Note: Steady State is not the same as DC Level
    - Need steady states before and after the sudden change
  - Transient: Always a negative exponential: $Ae^{-\frac{t}{\tau}}$
    - Time Constant: $\tau = RC$ or $\frac{L}{R}$ where R is the Thévenin resistance at the terminals of C or L
    - Find transient amplitude, A, from continuity since V_C or I_L cannot change instantly.
    - τ and A can also be found from the transfer function.

Steady States

A steady-state output assumes the input frequency, phase and amplitude are constant forever. You need to determine two $y_{SS}(t)$ steady state outputs: one for before the transient ($t < 0$) and one after ($t \geq 0$).
At $t = 0$, $y_{SS}(0-)$ means the first one and $y_{SS}(0+)$ means the second.



Method 1: Nodal analysis

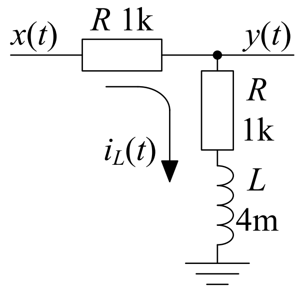
Input voltage is DC ($\omega = 0$)

$\Rightarrow Z_L = 0$ (for capacitor: $Z_C = \infty$)

So L acts as a short citcuit

Potential divider: $y_{SS} = \frac{1}{2}x$

$y_{SS}(0-) = 1$, $y_{SS}(0+) = 3$



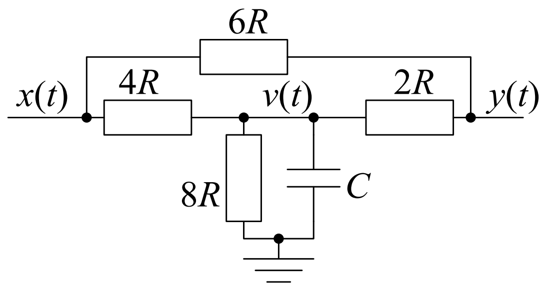
Method 2: Transfer function

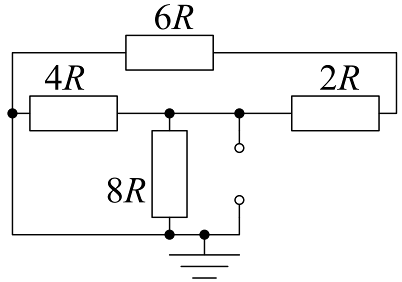
$\frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$

set $\omega = 0$: $\frac{Y}{X}(0) = \frac{1}{2}$

$y_{SS}(0-) = 1$, $y_{SS}(0+) = 3$

Sinusoidal input $\Rightarrow$ Sinusoidal steady state $\Rightarrow$ use phasors.
Then convert phasors to time waveforms to calculate the actual output voltages $y_{SS}(0-)$ and $y_{SS}(0+)$ at $t = 0$.





Method 1: Thévenin

(a) Remove the capacitor/inductor
(b) Set all sources to zero (including the input voltage source). Leave output unconnected.
(c) Calculate the Thévenin resistance between the capacitor/inductor terminals:
$R_{Th} = 8R||4R||(6R + 2R) = 2R$
(d) Time constant: $= R_{Th}C$ or $\frac{L}{R_{Th}}$
$\tau = R_{Th}C = 2RC$

Method 2: Transfer function

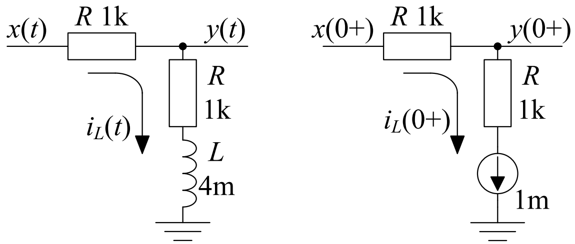
(a) Calculate transfer function using nodal analysis
KCL @ V: $\frac{V-X}{4R} + \frac{V}{8R} + j\omega CV + \frac{V-Y}{2R} = 0$
KCL @ Y: $\frac{Y-V}{2R} + \frac{Y-X}{6R} = 0$

$\rightarrow$ Eliminate V to get transfer Function: $\frac{Y}{X} = \frac{8j\omega RC+13}{32j\omega RC+16}$
(b) Time Constant $= \frac{1}{\text{Denominator corner frequency}}$
$\omega_d = \frac{16}{32RC} \Rightarrow \tau = \frac{1}{\omega_d} = 2RC$

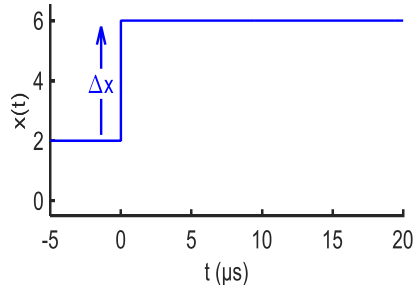
Determining Transient Amplitude

After an input change at $t = 0$, $y(t) = y_{SS}(t) + Ae^{-\frac{t}{\tau}}$.
$\Rightarrow y(0+) = y_{SS}(0+) + A \Rightarrow A = y(0+) - y_{SS}(0+)$
Method: (a) calculate true output $y(0+)$, (b) subtract $y_{SS}(0+)$ to get A

(i) Version 1: v_C or i_L continuity
$x(0-) = 2 \Rightarrow i_L(0-) = 1\,\text{mA}$
Continuity $\Rightarrow i_L(0+) = i_L(0-)$
Replace L with a $1\,\text{mA}$ current source
$y(0+) = x(0+) - iR = 6 - 1 = 5$

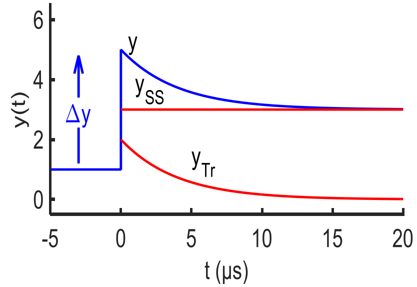


(i) Version 2: Transfer function
$H(j\omega) = \frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$
Input step, $\Delta x = x(0+) - x(0-) = +4$
$y(0+) = y(0-) + H(j\infty) \times \Delta x$
$= 1 + \Delta y = 1 + 1 \times 4 = 5$



(ii) $A = y(0+) - y_{SS}(0+) = 5 - 3 = 2$

(iii) $y(t) = y_{SS}(t) + Ae^{-t/\tau}$
$= 3 + 2e^{-t/2\mu}$

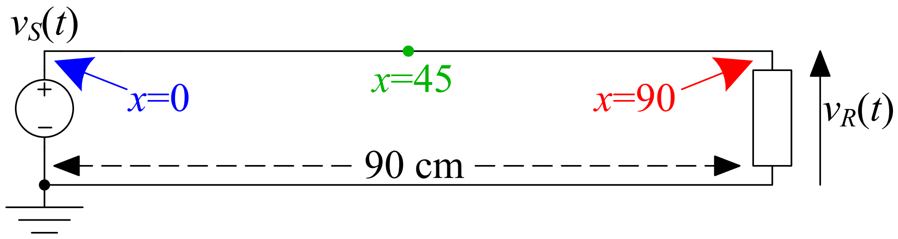


Transmission Line: constant L_0 and C_0 : inductance/capacitance per metre.

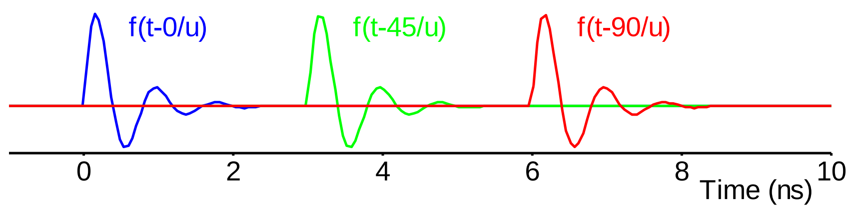
Forward wave travels along the line: $f_x(t) = f_0\left(t - \frac{x}{u}\right)$.

Velocity $u = \sqrt{\frac{1}{L_0 C_0}} \approx \frac{1}{2}c = 15\,\mathrm{cm/ns}$

$f_x(t)$ equals $f_0(t)$ but delayed by $\frac{x}{u}$.



Knowing $f_x(t)$ for $x = x_0$ fixes it for all other x.

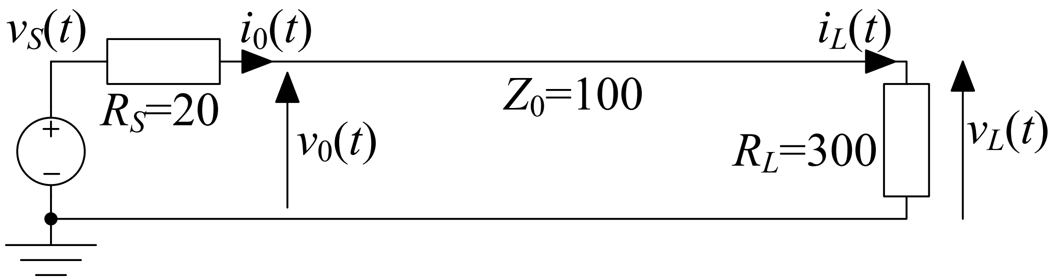


Backward wave: $g_x(t)$ is the same but travelling $\leftarrow$: $g_x(t) = g_0\left(t + \frac{x}{u}\right)$.

Voltage and current are: $v_x = f_x + g_x$ and $i_x = \frac{f_x - g_x}{Z_0}$ where i_x is positive in the $+x$ direction $(\rightarrow)$ and $Z_0 = \sqrt{\frac{L_0}{C_0}}$

Waveforms of f_x and g_x are determined by the connections at both ends.

Reflections



$$v_x = f_x + g_x$$
$$i_x = \frac{f_x - g_x}{Z_0}$$

At $x = L$, Ohm's law $\Rightarrow \frac{v_L(t)}{i_L(t)} = R_L \;\Rightarrow g_L(t) = \frac{R_L - Z_0}{R_L + Z_0} \times f_L(t)$.

Reflection coefficient: $\rho_L = \frac{g_L(t)}{f_L(t)} = \frac{R_L - Z_0}{R_L + Z_0}$

$\rho_L \in [-1, +1]$ and increases with R_L

Knowing $f_x(t)$ for $x = x_0$ now tells you $f_x,\ g_x,\ v_x,\ i_x\ \forall x$

At $x = 0$: $f_0(t) = \frac{Z_0}{R_S + Z_0} v_S(t) + \frac{R_S - Z_0}{R_S + Z_0} g_0(t) = \tau_0 v_S(t) + \rho_0 g_0(t)$

Wave bounces back and forth getting smaller with each reflection:

$$v_S(t) \xrightarrow{\times\tau_0} f_0(t) \xrightarrow{\times\rho_L} g_0(t + \tfrac{2L}{u}) \xrightarrow{\times\rho_0} f_0(t + \tfrac{2L}{u}) \xrightarrow{\times\rho_L} g_0(t + \tfrac{4L}{u}) \xrightarrow{\times\rho_0} \cdots$$

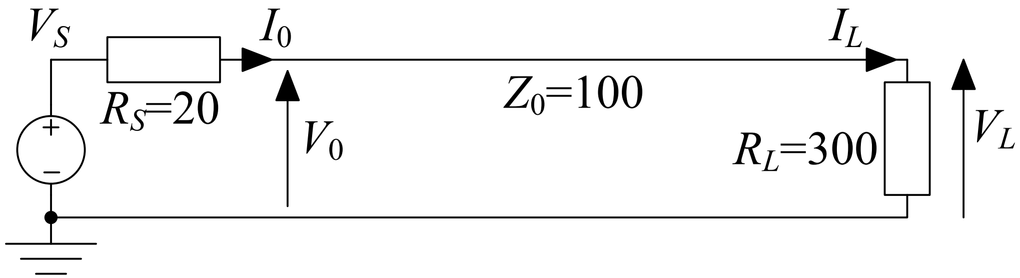
Infinite sum:

$$f_0(t) = \tau_0 v_S(t) + \tau_0 \rho_L \rho_0 v_S(t - \tfrac{2L}{u}) + \ldots = \textstyle\sum_{i=0}^{\infty} \tau_0 \rho_L^i \rho_0^i v_S\left(t - \frac{2Li}{u}\right)$$

Sinewaves are easier because:

1. Use phasors to eliminate t: $f_0(t) = A\cos(\omega t + \phi) \Leftrightarrow F_0 = Ae^{j\phi}$
2. Time delays are just phase shifts:
   $f_x(t) = A\cos\left(\omega\left(t - \frac{x}{u}\right) + \phi\right) \Leftrightarrow F_x = Ae^{j\left(\phi - \frac{\omega}{u}x\right)} = F_0 e^{-jkx}$
   $k = \frac{\omega}{u} = \frac{2\pi}{\lambda}$ is the wavenumber: radians per metre (c.f. ω in rad/s)

As before: $V_x = F_x + G_x$ and $I_x = \frac{F_x - G_x}{Z_0}$

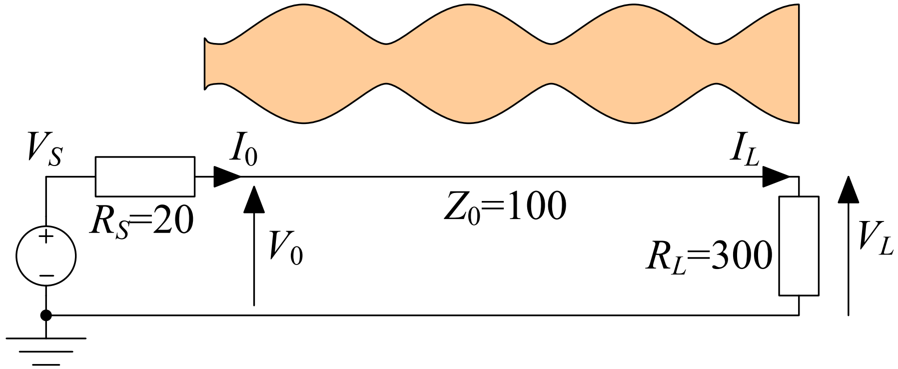


As before:
$G_L = \rho_L F_L$
$F_0 = \tau_0 V_S + \rho_0 G_0$

But $G_0 = F_0 \rho_L e^{-2jkL}$: roundtrip delay of $\frac{2L}{u}$ + reflection at $x = L$.
Substituting for G_0 in source end equation: $F_0 = \tau_0 V_S + \rho_0 F_0 \rho_L e^{-2jkL}$
$\Rightarrow F_0 = \frac{\tau_0}{1 - \rho_0 \rho_L \exp(-2jkL)} V_S$ so no infinite sums needed ☺

Standing waves arise whenever a wave meets its reflection:
at positions where the two waves are in phase their amplitudes add
but where they are anti-phase their amplitudes subtract.

At any point x,
delay of $\frac{x}{u} \Rightarrow$
$F_x = F_0 e^{-jkx}$



Backward wave: $G_x = \rho_L F_x e^{-2jk(L-x)}$: reflection + delay of $2\frac{L-x}{u}$
Voltage at x: $V_x = F_x + G_x = F_0 e^{-jkx}\left(1 + \rho_L e^{-2jk(L-x)}\right)$
Voltage Magnitude : $|V_x| = |F_0|\left|1 + \rho_L e^{-2jk(L-x)}\right|$: depends on x

If $\rho_L \geq 0$, max magnitude is $(1+\rho_L)|F_0|$ whenever $e^{-2jk(L-x)} = +1$
$\Rightarrow x = L$ or $x = L - \frac{\pi}{k}$ or $x = L - \frac{2\pi}{k}$ or ...

Min magnitude is $(1-\rho_L)|F_0|$ whenever $e^{-2jk(L-x)} = -1$
$\Rightarrow x = L - \frac{\pi}{2k}$ or $x = L - \frac{3\pi}{2k}$ or $x = L - \frac{5\pi}{2k}$ or ...