Revision Lecture 2: Transients & Lines

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Transients: Basic Ideas

- Transients happen in response to a sudden change
  
  - Input voltage/current abruptly changes its magnitude, frequency or phase
  
  - A switch alters the circuit

- 1st order circuits only: one capacitor/inductor

- All voltage/current waveforms are: Steady State + Transient

  - Steady States: find with nodal analysis or transfer function
    
    ▶ Note: Steady State is not the same as DC Level
    
    ▶ Need steady states before and after the sudden change

  - Transient: Always a negative exponential: \( Ae^{-\frac{t}{\tau}} \)
    
    ▶ Time Constant: \( \tau = RC \) or \( \frac{L}{R} \) where \( R \) is the Thévenin resistance at the terminals of \( C \) or \( L \)
    
    ▶ Find transient amplitude, \( A \), from continuity since \( V_C \) or \( I_L \) cannot change instantly.
    
    ▶ \( \tau \) and \( A \) can also be found from the transfer function.
A steady-state output assumes the input frequency, phase and amplitude are constant forever. You need to determine two $y_{SS}(t)$ steady state outputs: one for before the transient ($t < 0$) and one after ($t \geq 0$).
At $t = 0$, $y_{SS}(0−)$ means the first one and $y_{SS}(0+) = 3$ means the second.

Method 1: Nodal analysis
Input voltage is DC ($\omega = 0$)
$\Rightarrow Z_L = 0$ (for capacitor: $Z_C = \infty$)
So $L$ acts as a short circuit
Potential divider: $y_{SS} = \frac{1}{2} x$
$y_{SS}(0−) = 1$, $y_{SS}(0+) = 3$

Method 2: Transfer function
$\frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$
set $\omega = 0$: $\frac{Y}{X}(0) = \frac{1}{2}$
$y_{SS}(0−) = 1$, $y_{SS}(0+) = 3$

Sinusoidal input $\Rightarrow$ Sinusoidal steady state $\Rightarrow$ use phasors.
Then convert phasors to time waveforms to calculate the actual output voltages $y_{SS}(0−)$ and $y_{SS}(0+)$ at $t = 0$. 

\[ Y_x(j\omega) = \frac{R+j\omega L}{2R+j\omega L} \]

\[ Y_x(0) = \frac{1}{2} \]

\[ y_{SS}(0−) = 1, \quad y_{SS}(0+) = 3 \]
Method 1: Thévenin
(a) Remove the capacitor/inductor
(b) Set all sources to zero (including the input voltage source). Leave output unconnected.
(c) Calculate the Thévenin resistance between the capacitor/inductor terminals:
\[ R_{Th} = 8R||4R||(6R + 2R) = 2R \]
(d) Time constant: \( \tau = R_{Th}C = 2RC \)

Method 2: Transfer function
(a) Calculate transfer function using nodal analysis
KCL @ V: \( \frac{V-X}{4R} + \frac{V}{8R} + j\omega CV + \frac{V-Y}{2R} = 0 \)
KCL @ Y: \( \frac{Y-V}{2R} + \frac{Y-X}{6R} = 0 \)
\[ \rightarrow \text{Eliminate } V \text{ to get transfer Function: } \frac{Y}{X} = \frac{8j\omega RC + 13}{32j\omega RC + 16} \]
(b) Time Constant = \( \frac{1}{\text{Denominator corner frequency}} \)
\[ \omega_d = \frac{16}{32RC} \Rightarrow \tau = \frac{1}{\omega_d} = 2RC \]
Determining Transient Amplitude

After an input change at $t = 0$, $y(t) = y_{SS}(t) + Ae^{-t/\tau}$.

$\Rightarrow y(0+) = y_{SS}(0+) + A \Rightarrow A = y(0+) - y_{SS}(0+)$

Method: (a) calculate true output $y(0+)$, (b) subtract $y_{SS}(0+)$ to get $A$

(i) Version 1: $v_C$ or $i_L$ continuity

$x(0-) = 2 \Rightarrow i_L(0-) = 1 \text{mA}$

Continuity $\Rightarrow i_L(0+) = i_L(0-)$

Replace $L$ with a 1 mA current source

$y(0+) = x(0-) - iR = 6 - 1 = 5$

(ii) Transfer function

$H(j\omega) = \frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$

Input step, $\Delta x = x(0+) - x(0-) = +4$

$y(0+) = y(0-) + H(j\infty) \times \Delta x$

$= 1 + \Delta y = 1 + 1 \times 4 = 5$

(ii) $A = y(0+) - y_{SS}(0+) = 5 - 3 = 2$

(iii) $y(t) = y_{SS}(t) + Ae^{-t/\tau}$

$= 3 + 2e^{-t/2\mu}$
Transmission Line: constant $L_0$ and $C_0$: inductance/capacitance per metre.

Forward wave travels along the line: $f_x(t) = f_0 \left( t - \frac{x}{u} \right)$.

Velocity $u = \sqrt{\frac{1}{L_0C_0}} \approx \frac{1}{2} c = 15 \text{ cm/ns}$

$f_x(t)$ equals $f_0(t)$ but delayed by $\frac{x}{u}$.

Knowing $f_x(t)$ for $x = x_0$ fixes it for all other $x$.

Backward wave: $g_x(t)$ is the same but travelling $\leftarrow$: $g_x(t) = g_0 \left( t + \frac{x}{u} \right)$.

Voltage and current are: $v_x = f_x + g_x$ and $i_x = \frac{f_x - g_x}{Z_0}$ where $i_x$ is positive in the $+x$ direction ($\rightarrow$) and $Z_0 = \sqrt{\frac{L_0}{C_0}}$

Waveforms of $f_x$ and $g_x$ are determined by the connections at both ends.
At $x = L$, Ohm’s law $\Rightarrow \frac{v_L(t)}{i_L(t)} = R_L \Rightarrow g_L(t) = \frac{R_L-Z_0}{R_L+Z_0} \times f_L(t)$.

Reflection coefficient: $\rho_L = \frac{g_L(t)}{f_L(t)} = \frac{R_L-Z_0}{R_L+Z_0}$

$\rho_L \in [-1, +1]$ and increases with $R_L$

Knowing $f_x(t)$ for $x = x_0$ now tells you $f_x, g_x, v_x, i_x \forall x$

At $x = 0$: $f_0(t) = \frac{Z_0}{R_S+Z_0} v_S(t) + \frac{R_S-Z_0}{R_S+Z_0} g_0(t) = \tau_0 v_S(t) + \rho_0 g_0(t)$

Wave bounces back and forth getting smaller with each reflection:

$v_S(t) \xrightarrow{\times \tau_0} f_0(t) \xrightarrow{\times \rho_L} g_0(t + \frac{2L}{u}) \xrightarrow{\times \rho_0} f_0(t + \frac{2L}{u}) \xrightarrow{\times \rho_L} g_0(t + \frac{4L}{u}) \xrightarrow{\times \rho_0} \ldots$

Infinte sum:

$f_0(t) = \tau_0 v_S(t) + \tau_0 \rho_L \rho_0 v_S(t - \frac{2L}{u}) + \ldots = \sum_{i=0}^{\infty} \tau_0 \rho^i_L \rho^i_0 v_S \left( t - \frac{2Li}{u} \right)$
Sinewaves and Phasors

Sinewaves are easier because:

1. Use phasors to eliminate $t$: $f_0(t) = A \cos(\omega t + \phi) \Leftrightarrow F_0 = Ae^{j\phi}$

2. Time delays are just phase shifts:
   
   $f_x(t) = A \cos \left( \omega \left( t - \frac{x}{u} \right) + \phi \right) \Leftrightarrow F_x = Ae^{j(\phi - \frac{\omega}{u}x)} = F_0e^{-jkx}$

   $k = \frac{\omega}{u} = \frac{2\pi}{\lambda}$ is the wavenumber: radians per metre (c.f. $\omega$ in rad/s)

As before: $V_x = F_x + G_x$ and $I_x = \frac{F_x - G_x}{Z_0}$

As before:

$G_L = \rho_L F_L$

$F_0 = \tau_0 V_S + \rho_0 G_0$

But $G_0 = F_0\rho_L e^{-2jkL}$: roundtrip delay of $\frac{2L}{u} +$ reflection at $x = L$.

Substituting for $G_0$ in source end equation: $F_0 = \tau_0 V_S + \rho_0 F_0\rho_L e^{-2jkL}$

$\Rightarrow F_0 = \frac{\tau_0}{1 - \rho_0 \rho_L \exp(-2jkL)} V_S$ so no infinite sums needed 😊
**Standing Waves**

**Standing waves** arise whenever a wave meets its reflection: at positions where the two waves are in phase their amplitudes add but where they are anti-phase their amplitudes subtract.

At any point $x$, delay of $\frac{x}{u} \Rightarrow F_x = F_0 e^{-jkx}$

Backward wave: $G_x = \rho_L F_x e^{-2jk(L-x)}$: reflection + delay of $2\frac{L-x}{u}$

Voltage at $x$: $V_x = F_x + G_x = F_0 e^{-jkx} \left(1 + \rho_L e^{-2jk(L-x)}\right)$  

Voltage Magnitude: $|V_x| = |F_0| \left|1 + \rho_L e^{-2jk(L-x)}\right|$: depends on $x$

If $\rho_L \geq 0$, **max magnitude** is $(1 + \rho_L) |F_0|$ whenever $e^{-2jk(L-x)} = +1$  

$\Rightarrow x = L$ or $x = L - \frac{\pi}{k}$ or $x = L - \frac{2\pi}{k}$ or ...

**Min magnitude** is $(1 - \rho_L) |F_0|$ whenever $e^{-2jk(L-x)} = -1$  

$\Rightarrow x = L - \frac{\pi}{2k}$ or $x = L - \frac{3\pi}{2k}$ or $x = L - \frac{5\pi}{2k}$ or ...