

Revision Lecture 3:

Fre-

▷ quency Responses

Frequency Responses

Plotting Frequency Responses

LF and HF Asymptotes

Corner frequencies (linear factors)

Sketching Magnitude Responses (linear factors)

Sketching Phase response

Phase Response Example

Filters

Resonance

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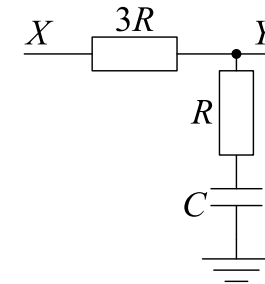
Phase Response Example

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- Frequency response is the ratio of two voltage/current phasors in a circuit: output phasor \div input phasor.
 - It is a complex number that depends on frequency.
 - Calculate using nodal analysis.
- For a linear circuit, the frequency response is always a ratio of two polynomials in $j\omega$ having real coefficients that depend on the component values.

$$\frac{Y}{X} = \frac{R + 1/j\omega C}{4R + 1/j\omega C} = \frac{j\omega RC + 1}{j\omega 4RC + 1}$$



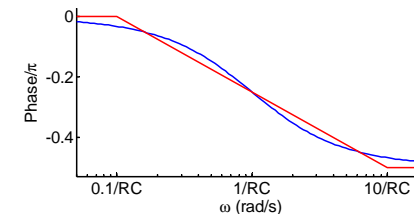
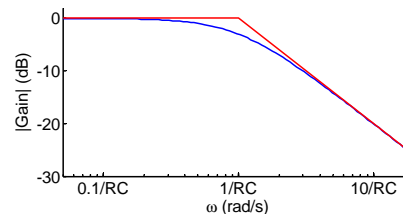
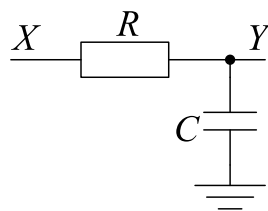
Plotting Frequency Responses

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- Plot the magnitude response and phase response as separate graphs. Use **log scale** for frequency and magnitude and **linear scale** for phase: this gives graphs that can be approximated by straight line segments.
- If $\frac{V_2}{V_1} = A(j\omega)^k = A \times j^k \times \omega^k$
 - magnitude is a **straight line with gradient k** :

$$\log \left| \frac{V_2}{V_1} \right| = \log |A| + k \log \omega$$
 - phase is a **constant $k \times \frac{\pi}{2}$** ($+\pi$ if $A < 0$):

$$\angle \left(\frac{V_2}{V_1} \right) = \angle A + k \angle j = \angle A + k \frac{\pi}{2}$$
- Measure magnitude using **decibels = $20 \log_{10} \frac{|V_2|}{|V_1|}$** .
 A gradient of k on log axes is the same as a gradient of $20k$ dB/decade ($\times 10$ in frequency) or $6k$ dB/octave ($\times 2$ in frequency).



$$\frac{Y}{X} = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} = \frac{1}{\frac{j\omega}{\omega_c} + 1} \quad \text{where } \omega_c = \frac{1}{RC}$$

LF and HF Asymptotes

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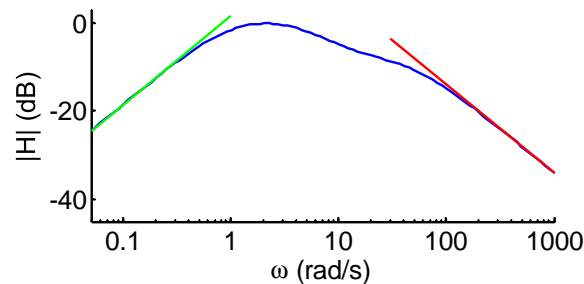
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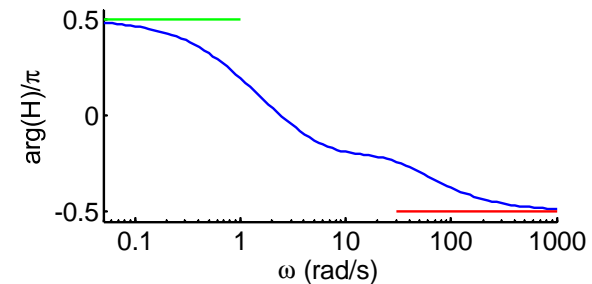
Resonance

- Frequency response is always a ratio of two polynomials in $j\omega$ with real coefficients that depend on the component values.
 - The terms with the **lowest** power of $j\omega$ on top and bottom gives the **low-frequency** asymptote
 - The terms with the **highest** power of $j\omega$ on top and bottom gives the **high-frequency** asymptote.

$$\text{Example: } H(j\omega) = \frac{60(j\omega)^2 + 720(j\omega)}{3(j\omega)^3 + 165(j\omega)^2 + 762(j\omega) + 600}$$



$$\text{LF: } H(j\omega) \simeq 1.2j\omega$$
$$\text{HF: } H(j\omega) \simeq 20(j\omega)^{-1}$$



Corner frequencies (linear factors)

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- We can factorize the numerator and denominator into linear terms of the form $(aj\omega + b) \simeq \begin{cases} b & \omega < \left| \frac{b}{a} \right| \\ aj\omega & \omega > \left| \frac{b}{a} \right| \end{cases}$.
- At the corner frequency, $\omega_c = \left| \frac{b}{a} \right|$ the slope of the magnitude response changes by ± 1 ($\equiv \pm 20$ dB/decade) because the linear term introduces another factor of ω into the numerator or denominator for $\omega > \omega_c$.
- The phase changes by $\pm \frac{\pi}{2}$ because the linear term introduces another factor of j into the numerator or denominator for $\omega > \omega_c$.
 - The phase change is **gradual** and takes place over the range $0.1\omega_c$ to $10\omega_c$ ($\pm \frac{\pi}{2}$ spread over two decades in ω).
- When a and b are real and positive, it is often convenient to write $(aj\omega + b) = b \left(\frac{j\omega}{\omega_c} + 1 \right)$.
- The corner frequencies are the absolute values of the roots (values of $j\omega$) of the numerator and denominator polynomials.

Sketching Magnitude Responses (linear factors)

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1. Find corner frequencies: (a) factorize the numerator/denominator polynomials or (b) find their roots
2. Find LF and HF asymptotes. $A(j\omega)^k$ has a slope of k ; substitute the value of ω_c to get the value at first/last corner frequency.
3. At a corner frequency, the gradient of the magnitude response changes by ± 1 (± 20 dB/decade). $+$ for numerator (top line) and $-$ for denominator (bottom line).
4. $|H(j\omega_2)| = \left(\frac{\omega_2}{\omega_1}\right)^k |H(j\omega_1)|$ if the gradient between them is k .

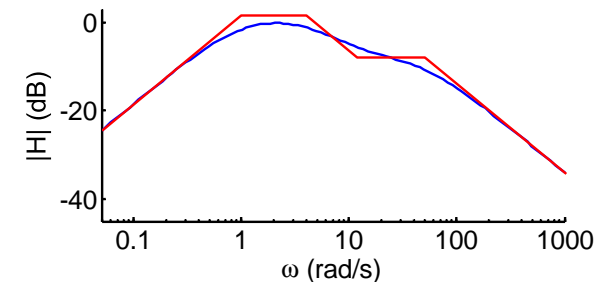
$$H(j\omega) = 1.2 \frac{j\omega \left(\frac{j\omega}{12} + 1\right)}{\left(\frac{j\omega}{1} + 1\right) \left(\frac{j\omega}{4} + 1\right) \left(\frac{j\omega}{50} + 1\right)}$$

$$\text{LF: } 1.2j\omega \Rightarrow |H(j1)| = 1.2 \text{ (1.6 dB)}$$

$$|H(j4)| = \left(\frac{4}{1}\right)^0 \times 1.2 = 1.2$$

$$|H(j12)| = \left(\frac{12}{4}\right)^{-1} \times 1.2 = 0.4$$

$$|H(j50)| = \left(\frac{50}{12}\right)^0 \times 0.4 = 0.4 \text{ (-8 dB)}. \text{ As a check: HF: } 20(j\omega)^{-1}$$



Sketching Phase response

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- Phase response is more complicated than the magnitude response:
 - (a) there are twice as many corner frequencies
 - (b) negative factors add an extra π onto the phase
 - (c) Phases wrap around: $2\pi \equiv 0$.
- A corner in the magnitude response at ω_c gives rise to phase response corners at $0.1\omega_c$ and $10\omega_c$. The gradient change at each corner is $\pm \frac{\pi}{4}$ radians per decade (\Rightarrow gradient = $m \times \frac{\pi}{4}$).
 - For a factor $(aj\omega + b)$ with $\frac{b}{a} > 0$ (normally true) the first change is in the same direction as for the magnitude response
 - ▷ a numerator factor has a positive gradient change at $\omega = 0.1 \left| \frac{b}{a} \right|$ and negative at $\omega = 10 \left| \frac{b}{a} \right|$.
 - The signs of the gradient changes are reversed if $\frac{b}{a} < 0$ and reversed again for the denominator.
- LF asymptote = $A(j\omega)^k \Rightarrow$ LF phase = $k\frac{\pi}{2}$ ($+\pi$ if $A < 0$)
- When slope is $m\frac{\pi}{4}$: $\angle H(j\omega_2) = m\frac{\pi}{4} \times \log_{10} \left(\frac{\omega_2}{\omega_1} \right) + \angle H(j\omega_1)$

Phase Response Example

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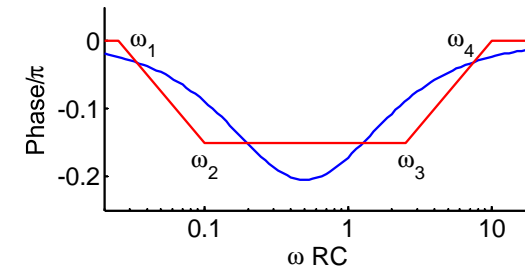
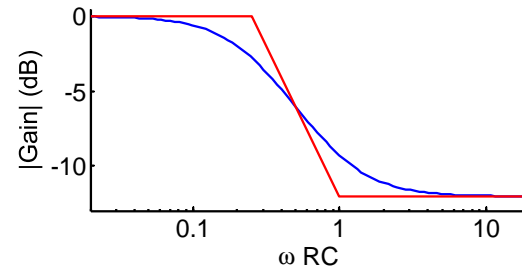
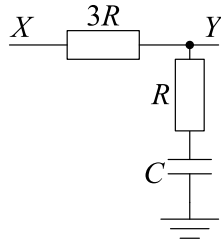
Phase Response

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$$\frac{Y}{X} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}} = \frac{j\omega RC + 1}{4j\omega RC + 1} \Rightarrow \omega_D = \frac{1}{4RC} = \frac{0.25}{RC}, \omega_N = \frac{1}{RC}$$



Phase Response: (all gradient changes $\Delta = \pm \frac{\pi}{4}$ /decade)

Denominator $\omega_D = \frac{0.25}{RC} \Rightarrow \Delta = -$ at $\omega_1 = \frac{0.025}{RC}$ and $+$ at $\omega_3 = \frac{2.5}{RC}$.

Numerator $\omega_N = \frac{1}{RC} \Rightarrow \Delta = +$ at $\omega_2 = \frac{0.1}{RC}$ and $-$ at $\omega_4 = \frac{10}{RC}$.

Corner Values

LF asymptote: $H(j\omega) = 1 \Rightarrow \angle H(j\omega_1) = 0$

$$\angle H(j\omega_2) = \angle H(j\omega_1) - \frac{\pi}{4} \log_{10} \left(\frac{\omega_2}{\omega_1} \right) = -\frac{\pi}{4} \log_{10} (4) = -0.151\pi$$

$$\angle H(j\omega_3) = \angle H(j\omega_2) - 0 \times \log_{10} \left(\frac{\omega_3}{\omega_2} \right) = -0.151\pi$$

$$\angle H(j\omega_4) = \angle H(j\omega_3) + \frac{\pi}{4} \log_{10} \left(\frac{\omega_4}{\omega_3} \right) = -0.151\pi + \frac{\pi}{4} \log_{10} (4) = 0$$

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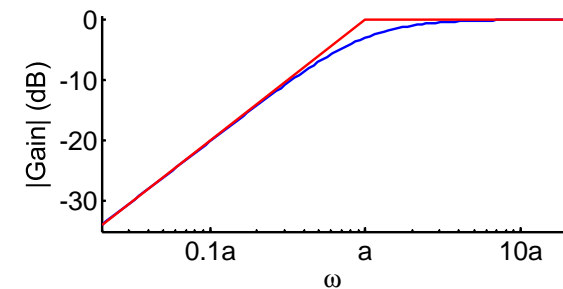
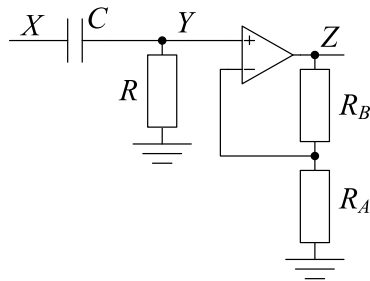
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Phase Response Example

▷ Filters

Resonance

- Filter: a circuit designed to amplify some frequencies and/or attenuate others. Very widely used.
- The **order** of the filter is the highest power of $j\omega$ in the denominator of the frequency response.
- Often use op-amps to provide a low impedance output.



$$\frac{Y}{X} = \frac{R}{R + 1/j\omega C} = \frac{j\omega RC}{j\omega RC + 1} = \frac{j\omega RC}{\frac{j\omega}{a} + 1}$$

$$\frac{Z}{X} = \left(1 + \frac{R_B}{R_A}\right) \times \frac{j\omega RC}{\frac{j\omega}{a} + 1}$$

Resonance

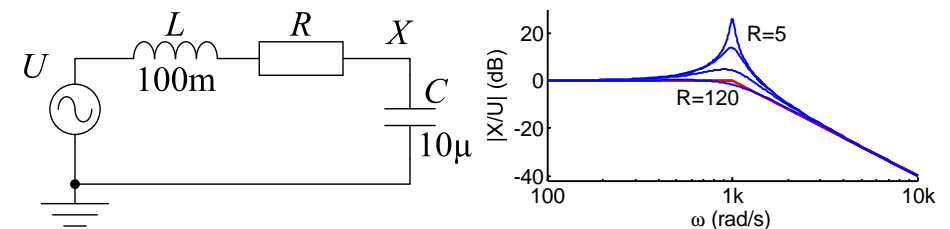
- Resonant circuits have quadratic factors that cannot be factorized
 - $H(j\omega) = a(j\omega)^2 + bj\omega + c = c \left(\left(\frac{j\omega}{\omega_0} \right)^2 + 2\zeta \left(\frac{j\omega}{\omega_0} \right) + 1 \right)$
 - Resonant frequency:** $\omega_0 = \sqrt{\frac{c}{a}}$ determines the horizontal position
 - Damping Factor:** $\zeta = \frac{b\omega_0}{2c} = \frac{b}{\sqrt{4ac}}$ determines the response shape
 - Equivalently **Quality Factor:** $Q \triangleq \frac{\omega \times \text{Stored Energy}}{\text{Power Dissipation}} \approx \frac{1}{2\zeta} = \frac{c}{b\omega_0}$
- At $\omega = \omega_0$, outer terms cancel ($a(j\omega)^2 = -c$): $\Rightarrow H(j\omega) = jb\omega_0 = 2jc\zeta$
 - $|H(j\omega_0)| = 2\zeta$ times the straight line approximation at ω_0 .
 - 3 dB bandwidth of peak $\simeq 2\zeta\omega_0 \approx \frac{\omega_0}{Q}$. $\Delta\text{phase} = \pm\pi$ over 2ζ decades

$$R = 5, 20, 60, 120$$

$$\zeta = \frac{1}{40}, \frac{1}{10}, \frac{3}{10}, \frac{6}{10}$$

$$Q = \frac{|Z_C(\omega_0) \text{ or } Z_L(\omega_0)|}{R} = 20, 5, \frac{5}{3}, \frac{5}{6}$$

$$\frac{\text{PeakGain}}{\text{CornerGain}} = \frac{1}{2\zeta} \approx Q$$



$$\frac{X}{U} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}, \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}}, \quad Q = \frac{\omega_0 L}{R} = \frac{1}{2\zeta}$$