Revision Lecture 1: Nodal Analysis & Frequency Responses

Exam Format

Question 1 (40%): eight short parts covering the whole syllabus.

Questions 2 and 3: single topic questions (answer both)

Syllabus

Does include: Everything in the notes.

Does not include: Two-port parameters (2008:1j), Gaussian elimination (2007:2c), Application areas (2008:3d), Nullators and Norators (2008:4c), Small-signal component models (2008:4e), Gain-bandwidth product (2006:4c), Zener Diodes (2008/9 syllabus), Non-ideal models of L, C and transformer (2011/12 syllabus), Transmission lines VSWR and crank diagram (2011/12 syllabus).

(1) Pick reference node.

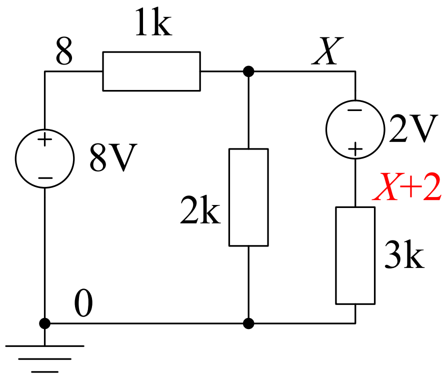
(2) Label nodes: 8, X and $X+2$ since it is joined to X via a voltage source.

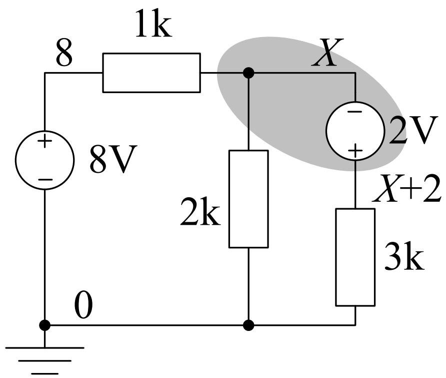
(3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "super-node" giving one equation

$$\frac{X-8}{1}+\frac{X}{2}+\frac{(X+2)-0}{3}=0$$

Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

(4) Solve the equations: $X=4$

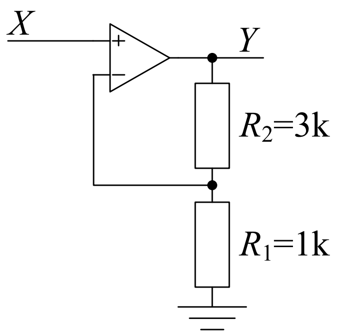




- Ideal Op Amp: (a) Zero input current, (b) Infinite gain
  (b) $\Rightarrow V_+ = V_-$ provided the circuit has negative feedback.
- Negative Feedback: An increase in V_{out} makes $(V_+ - V_-)$ decrease.

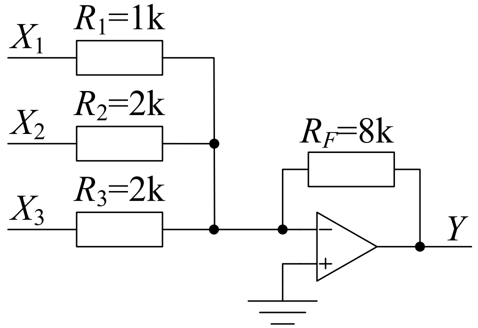
Non-inverting amplifier

$$Y = \left(1 + \tfrac{3}{1}\right) X$$



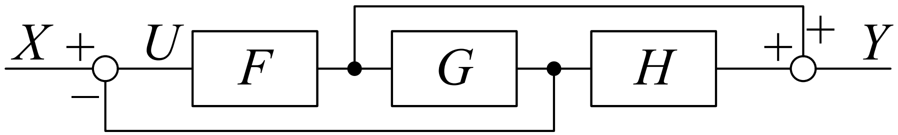
Inverting amplifier

$$Y = \tfrac{-8}{1} X_1 + \tfrac{-8}{2} X_2 + \tfrac{-8}{2} X_3$$



Nodal Analysis: Use two separate KCL equations at V_+ and V_-.
Do not do KCL at V_{out} except to find the op-amp output current.

Blocks are labelled with their gains and connected using adder/subtractors shown as circles. Adder inputs are marked + for add or − for subtract.



To analyse:

1. Label the inputs, the outputs and the output of each adder.
2. Write down an equation for each variable:
   - $U = X - FGU$, $\quad Y = FU + FGHU$
   - Follow signals back though the blocks until you meet a labelled node.
3. Solve the equations (eliminate intermediate node variables):
   - $U(1 + FG) = X \quad \Rightarrow \quad U = \frac{X}{1+FG}$
   - $Y = (1 + GH)FU = \frac{(1+GH)F}{1+FG}X$

[Note: "Wires" carry information not current: KCL not valid]

Diodes

Each diode in a circuit is in one of two modes; each has an equality condition and an inequality condition:

- ☐ Off: $I_D = 0,\ V_D < 0.7 \quad \Rightarrow \quad$ Diode = open circuit
- ☐ On: $V_D = 0.7,\ I_D > 0 \quad \Rightarrow \quad$ Diode = $0.7\,\text{V}$ voltage source

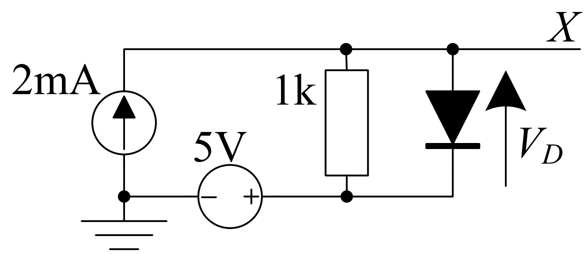
(a) Guess the mode
(b) Do nodal analysis assuming the equality condition
(c) Check the inequality condition. If the inequality condition fails, you made the wrong guess.

- Assume Diode Off

$$X = 5 + 2 = 7$$
$$V_D = 2 \quad \text{Fail: } V_D > 0.7$$

- Assume Diode On

$$X = 5 + 0.7 = 5.7$$
$$I_D + \frac{0.7}{1\,\text{k}} = 2\,\text{mA} \quad \text{OK: } I_D > 0$$

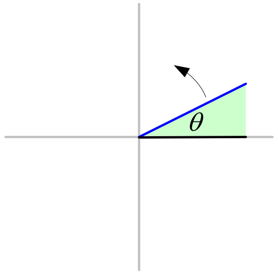


Reactive Components

- Impedances: R, $j\omega L$, $\frac{1}{j\omega C} = \frac{-j}{\omega C}$.
  - Admittances: $\frac{1}{R}$, $\frac{1}{j\omega L} = \frac{-j}{\omega L}$, $j\omega C$
- In a capacitor or inductor, the Current and Voltage are $90°$ apart :
  - CIVIL: Capacitor - I leads V; Inductor - I lags V
- Average current (or DC current) through a capacitor is always zero
- Average voltage across an inductor is always zero
- Average power absorbed by a capacitor or inductor is always zero

A phasor represents a time-varying sinusoidal waveform by a fixed complex number.

Waveform	Phasor	
$x(t) = F\cos\omega t - G\sin\omega t$	$X = F + jG$	[Note minus sign]
$x(t) = A\cos(\omega t + \theta)$	$X = Ae^{j\theta} = A\angle\theta$	
$\max(x(t)) = A$	$\lvert X\rvert = A$	



$x(t)$ is the projection of a rotating rod onto the real (horizontal) axis.

$X = F + jG$ is its starting position at $t = 0$.

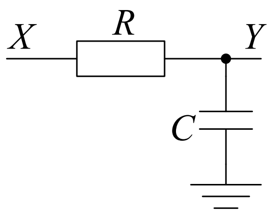
RMS Phasor: $\widetilde{V} = \frac{1}{\sqrt{2}}V \quad \Rightarrow \quad \left|\widetilde{V}\right|^2 = \langle x^2(t)\rangle$

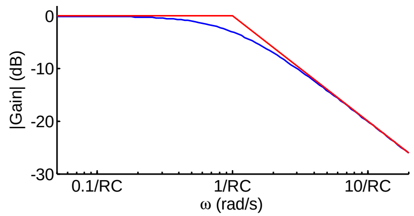
Complex Power: $\widetilde{V}\widetilde{I}^* = |\widetilde{I}|^2 Z = \frac{|\widetilde{V}|^2}{Z^*} = P + jQ$

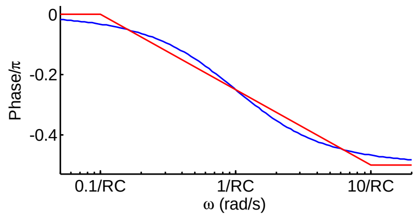
P is average power (Watts), Q is reactive power (VARs)

Plotting Frequency Responses

- Plot the magnitude response and phase response as separate graphs. Use log scale for frequency and magnitude and linear scale for phase: this gives graphs that can be approximated by straight line segments.
- If $\frac{V_2}{V_1} = A\,(j\omega)^k = A \times j^k \times \omega^k$ (where A is real)
  - magnitude is a straight line with gradient k:
  $$\log\left|\frac{V_2}{V_1}\right| = \log|A| + k\log\omega$$
  - phase is a constant $k \times \frac{\pi}{2}$ ($+\pi$ if $A < 0$):
  $$\angle\left(\frac{V_2}{V_1}\right) = \angle A + k\angle j = \angle A + k\frac{\pi}{2}$$
- Measure magnitude response using decibels $= 20\log_{10}\frac{|V_2|}{|V_1|}$. A gradient of k on log axes is equivalent to $20k\,\text{dB}/\text{decade}$ ($\times 10$ in frequency) or $6k\,\text{dB}/\text{octave}$ ($\times 2$ in frequency).





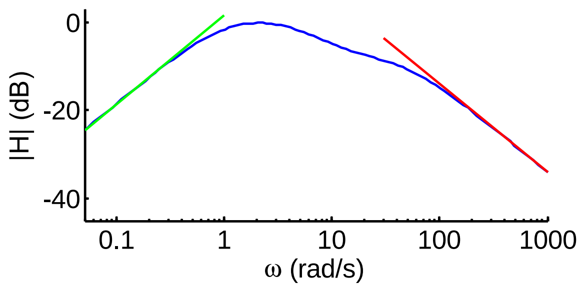


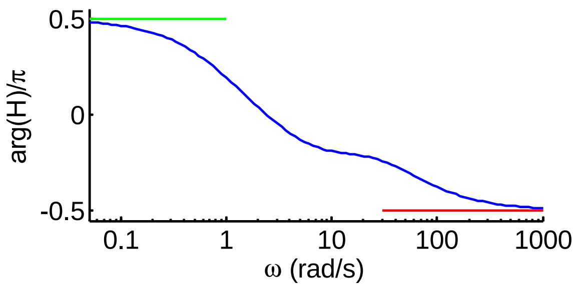
$$\frac{Y}{X} = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{1}{j\omega RC+1} = \frac{1}{\frac{j\omega}{\omega_c}+1} \text{ where } \omega_c = \frac{1}{RC}$$

LF and HF Asymptotes

- Frequency response is always a ratio of two polynomials in $j\omega$ with real coefficients that depend on the component values.
  - The terms with the lowest power of $j\omega$ on top and bottom gives the low-frequency asymptote
  - The terms with the highest power of $j\omega$ on top and bottom gives the high-frequency asymptote.

Example: $H(j\omega) = \frac{60(j\omega)^2+720(j\omega)}{3(j\omega)^3+165(j\omega)^2+762(j\omega)+600}$





LF: $H(j\omega) \simeq 1.2j\omega$
HF: $H(j\omega) \simeq 20\,(j\omega)^{-1}$

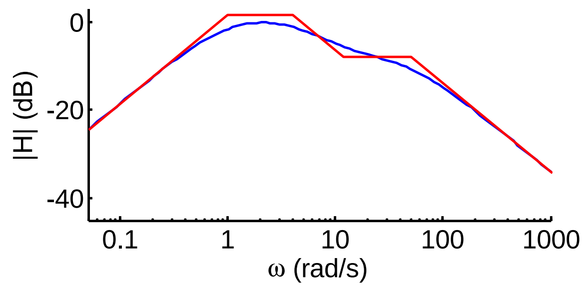
Corner frequencies (linear factors)

- We can factorize the numerator and denominator into linear terms of the form $(aj\omega + b) \simeq \begin{cases} b & \omega < \left|\frac{b}{a}\right| \\ aj\omega & \omega > \left|\frac{b}{a}\right| \end{cases}$.
- At the corner frequency, $\omega_c = \left|\frac{b}{a}\right|$, the slope of the magnitude response changes by ± 1 (± 20 dB/decade) because the linear term introduces another factor of ω into the numerator or denominator for $\omega > \omega_c$.
- The phase changes by $\pm\frac{\pi}{2}$ because the linear term introduces another factor of j into the numerator or denominator for $\omega > \omega_c$.
  - The phase change is gradual and takes place over the range $0.1\omega_c$ to $10\omega_c$ ($\pm\frac{\pi}{2}$ spread over two decades in ω).
- When a and b are real and positive, it is often convenient to write $(aj\omega + b) = b\left(\frac{j\omega}{\omega_c} + 1\right)$.
- The corner frequencies are the absolute values of the roots of the numerator and denominator polynomials (values of $j\omega$).

Sketching Magnitude Responses (linear factors)

1. Find corner frequencies: (a) factorize the numerator/denominator polynomials or (b) find their roots
2. Find LF and HF asymptotes. $A\,(j\omega)^k$ has a slope of k; substitute $\omega = \omega_c$ to get the response at first/last corner frequency.
3. At a corner frequency, the gradient of the magnitude response changes by ± 1 (± 20 dB/decade). $+$ for numerator (top line) and $-$ for denominator (bottom line).
4. $|H(j\omega_2)| = \left(\frac{\omega_2}{\omega_1}\right)^k |H(j\omega_1)|$ if the gradient between them is k.

$$H(j\omega) = 1.2\frac{j\omega\left(\frac{j\omega}{12}+1\right)}{\left(\frac{j\omega}{1}+1\right)\left(\frac{j\omega}{4}+1\right)\left(\frac{j\omega}{50}+1\right)}$$



LF: $1.2j\omega \Rightarrow |H(j1)| = 1.2\,(1.6\,\text{dB})$

$|H(j4)| = \left(\frac{4}{1}\right)^0 \times 1.2 = 1.2$

$|H(j12)| = \left(\frac{12}{4}\right)^{-1} \times 1.2 = 0.4$

$|H(j50)| = \left(\frac{50}{12}\right)^0 \times 0.4 = 0.4\,(-8\,\text{dB})$. As a check: HF: $20\,(j\omega)^{-1}$

[Sketching Responses (linear factors): Summary]

LF and HF asymptotes

The LF and HF asymptotes give you both the *magnitude* and *phase* at very low and very high frequencies. The LF asymptote is found by taking the terms with the lowest power of ω in numerator and denominator; the HF asymptote is found by taking the terms with the highest power of ω.

Magnitude response

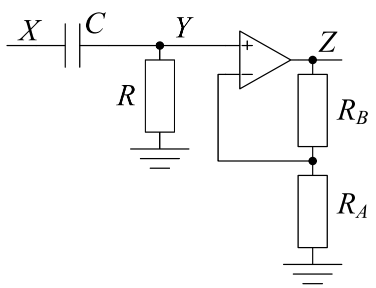
The corner frequency for a linear factor $(aj\omega + b)$ is at $\omega_c = \left|\frac{b}{a}\right|$. At each corner frequency, the slope of the magnitude response changes by $\pm 6\,\mathrm{dB/octave}$ ($= \pm 20\,\mathrm{dB/decade}$). The change is $+$ve for numerator corner frequencies and $-$ve for denominator corner frequencies. An octave is a factor of 2 in frequency and a decade is a factor of 10 in frequency. The number of decades between ω_1 and ω_2 is given by $\log_{10} \frac{\omega_2}{\omega_1}$.

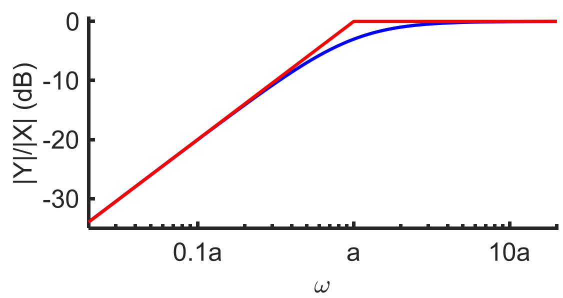
Phase Response

For each corner frequency, ω_c, the slope of the phase response changes *twice*: once at $0.1\omega_c$ and once, in the opposite direction, at $10\omega_c$. The change in slope is always $\pm 0.25\pi\,\mathrm{rad/decade}$. If a and b have the same sign (normal case), then the first slope change (at $0.1\omega_c$) is in the same direction as that of the magnitude response ($+$ve for numerator and $-$ve for denominator); if a and b have opposite signs (rare), then the sign of the slope change is reversed. The second slope change (at $10\omega_c$) always has the opposite sign from the first.

Filters

- Filter: a circuit designed to amplify some frequencies and/or attenuate others. Very widely used.
- The order of the filter is the highest power of $j\omega$ in the denominator of the frequency response.
- Often use op-amps to provide a low impedance output.





$$\frac{Y}{X} = \frac{R}{R + {}^1/_{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1} = \frac{j\omega RC}{\frac{j\omega}{a} + 1}$$

$$\frac{Z}{X} = \frac{Z}{Y} \times \frac{Y}{X} = \left(1 + \frac{R_B}{R_A}\right) \times \frac{j\omega RC}{\frac{j\omega}{a} + 1}$$

Resonance

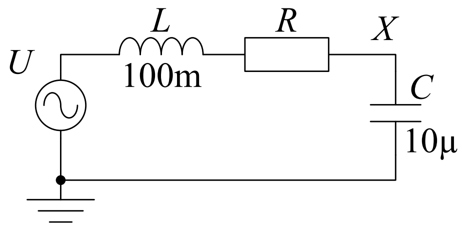
- Resonant circuits have quadratic factors that cannot be factorized
  - $H(j\omega) = a(j\omega)^2 + bj\omega + c = c\left(\left(\frac{j\omega}{\omega_0}\right)^2 + 2\zeta\left(\frac{j\omega}{\omega_0}\right) + 1\right)$
  - Corner frequency: $\omega_0 = \sqrt{\frac{c}{a}}$ determines the horizontal position
  - Damping Factor: $\zeta = \frac{b\omega_0}{2c} = \frac{b}{\sqrt{4ac}}$ determines the response shape
  - Equivalently Quality Factor: $Q \triangleq \frac{\omega \times \text{Average Stored Energy}}{\text{Average Power Dissipation}} \approx \frac{1}{2\zeta} = \frac{c}{b\omega_0}$
- At $\omega = \omega_0$, outer terms cancel $\left(a(j\omega)^2 = -c\right)$: $\Rightarrow H(j\omega) = jb\omega_0 = 2jc\zeta$
  - $|H(j\omega_0)| = 2\zeta$ times the straight line approximation at ω_0.
  - 3 dB bandwidth of peak $\simeq 2\zeta\omega_0 \approx \frac{\omega_0}{Q}$. $\Delta\text{phase} = \pm\pi$ over 2ζ decades

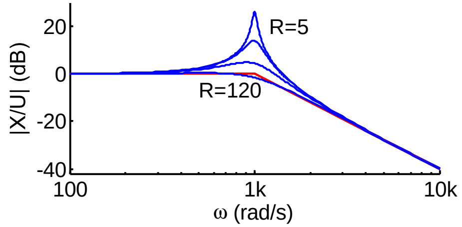
$R = 5, 20, 60, 120$

$\zeta = \frac{1}{40}, \frac{1}{10}, \frac{3}{10}, \frac{6}{10}$

$Q = \frac{|Z_C(\omega_0) \text{ or } Z_L(\omega_0)|}{R} = 20, 5, \frac{5}{3}, \frac{5}{6}$

$\frac{\text{Gain@}\,\omega_0}{\text{CornerGain}} = \frac{1}{2\zeta} \approx Q$





$\frac{X}{U} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$

$\omega_0 = \sqrt{\frac{1}{LC}}, \ \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}, \ Q = \frac{\omega_0 L}{R} = \frac{1}{2\zeta}$

Revision Lecture 2: Transients & Lines

- Transients happen in response to a sudden change
  - Input voltage/current abruptly changes its magnitude, frequency or phase
  - A switch alters the circuit
- 1st order circuits only: one capacitor/inductor
- All voltage/current waveforms are: Steady State + Transient
  - Steady States: find with nodal analysis or transfer function
    - Note: Steady State is not the same as DC Level
    - Need steady states before and after the sudden change
  - Transient: Always a negative exponential: $Ae^{-\frac{t}{\tau}}$
    - Time Constant: $\tau = RC$ or $\frac{L}{R}$ where R is the Thévenin resistance at the terminals of C or L
    - Find transient amplitude, A, from continuity since V_C or I_L cannot change instantly.
    - τ and A can also be found from the transfer function.

A steady-state output assumes the input frequency, phase and amplitude are constant forever. You need to determine two $y_{SS}(t)$ steady state outputs: one for before the transient ($t < 0$) and one after ($t \geq 0$).
At $t = 0$, $y_{SS}(0-)$ means the first one and $y_{SS}(0+)$ means the second.

Method 1: Nodal analysis

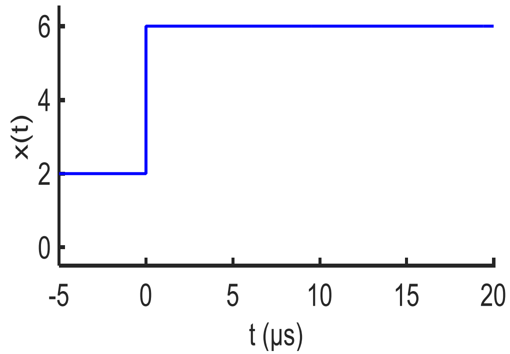
Input voltage is DC ($\omega = 0$)

$\Rightarrow Z_L = 0$ (for capacitor: $Z_C = \infty$)

So L acts as a short citcuit

Potential divider: $y_{SS} = \frac{1}{2}x$

$y_{SS}(0-) = 1$, $y_{SS}(0+) = 3$

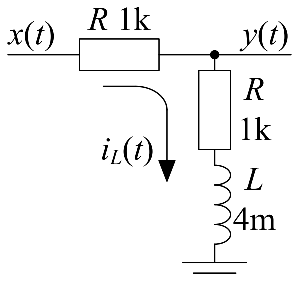


Method 2: Transfer function

$\frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$

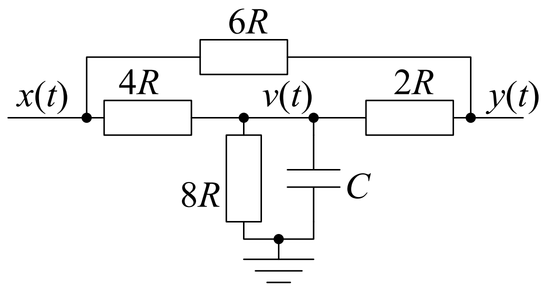
set $\omega = 0$: $\frac{Y}{X}(0) = \frac{1}{2}$

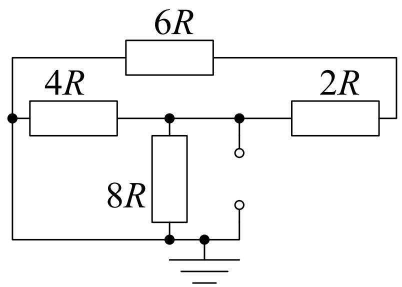
$y_{SS}(0-) = 1$, $y_{SS}(0+) = 3$



Sinusoidal input ⇒ Sinusoidal steady state ⇒ use phasors.
Then convert phasors to time waveforms to calculate the actual output voltages $y_{SS}(0-)$ and $y_{SS}(0+)$ at $t = 0$.

Determining Time Constant





Method 1: Thévenin

(a) Remove the capacitor/inductor

(b) Set all sources to zero (including the input voltage source). Leave output unconnected.

(c) Calculate the Thévenin resistance between the capacitor/inductor terminals:

$$R_{Th} = 8R||4R||(6R + 2R) = 2R$$

(d) Time constant: $= R_{Th}C$ or $\frac{L}{R_{Th}}$

$$\tau = R_{Th}C = 2RC$$

Method 2: Transfer function

(a) Calculate transfer function using nodal analysis

KCL @ V: $\frac{V-X}{4R} + \frac{V}{8R} + j\omega CV + \frac{V-Y}{2R} = 0$

KCL @ Y: $\frac{Y-V}{2R} + \frac{Y-X}{6R} = 0$

$\rightarrow$ Eliminate V to get transfer Function: $\frac{Y}{X} = \frac{8j\omega RC+13}{32j\omega RC+16}$

(b) Time Constant $= \frac{1}{\text{Denominator corner frequency}}$

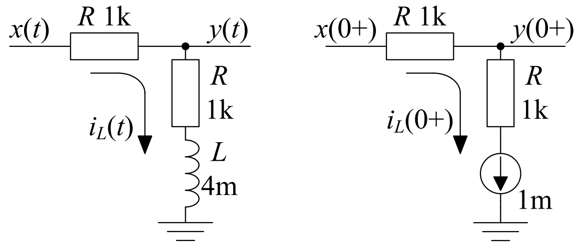
$$\omega_d = \frac{16}{32RC} \Rightarrow \tau = \frac{1}{\omega_d} = 2RC$$

Determining Transient Amplitude

After an input change at $t = 0$, $y(t) = y_{SS}(t) + Ae^{-\frac{t}{\tau}}$.
$\Rightarrow y(0+) = y_{SS}(0+) + A \Rightarrow A = y(0+) - y_{SS}(0+)$
Method: (a) calculate true output $y(0+)$, (b) subtract $y_{SS}(0+)$ to get A

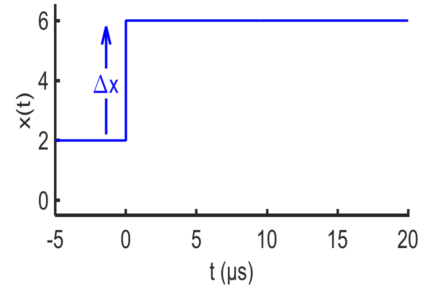
(i) Version 1: v_C or i_L continuity

$x(0-) = 2 \Rightarrow i_L(0-) = 1\,\text{mA}$
Continuity $\Rightarrow i_L(0+) = i_L(0-)$
Replace L with a $1\,\text{mA}$ current source
$y(0+) = x(0+) - iR = 6 - 1 = 5$



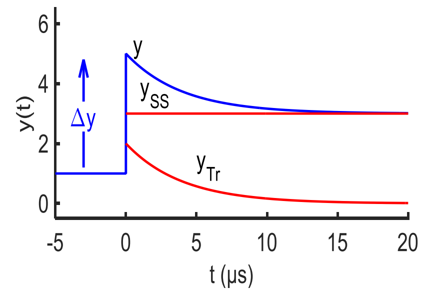
(i) Version 2: Transfer function

$H(j\omega) = \frac{Y}{X}(j\omega) = \frac{R+j\omega L}{2R+j\omega L}$
Input step, $\Delta x = x(0+) - x(0-) = +4$
$y(0+) = y(0-) + H(j\infty) \times \Delta x$
$= 1 + \Delta y = 1 + 1 \times 4 = 5$



(ii) $A = y(0+) - y_{SS}(0+) = 5 - 3 = 2$

(iii) $y(t) = y_{SS}(t) + Ae^{-t/\tau}$
$= 3 + 2e^{-t/2\mu}$

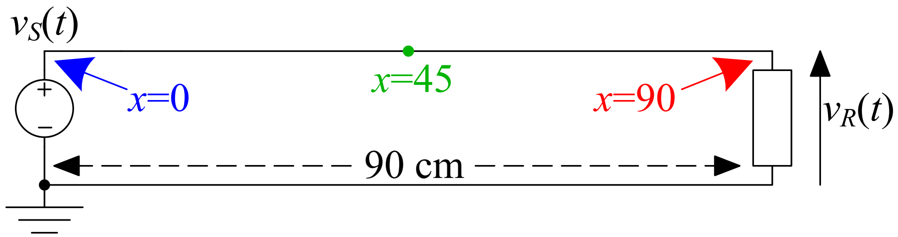


Transmission Line: constant L_0 and C_0 : inductance/capacitance per metre.

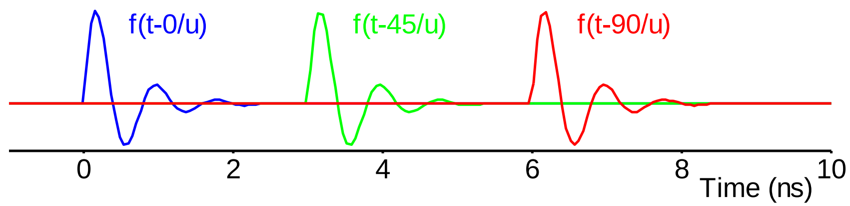
Forward wave travels along the line: $f_x(t) = f_0\left(t - \frac{x}{u}\right)$.

Velocity $u = \sqrt{\frac{1}{L_0 C_0}} \approx \frac{1}{2}c = 15\,\text{cm}/\text{ns}$

$f_x(t)$ equals $f_0(t)$ but delayed by $\frac{x}{u}$.



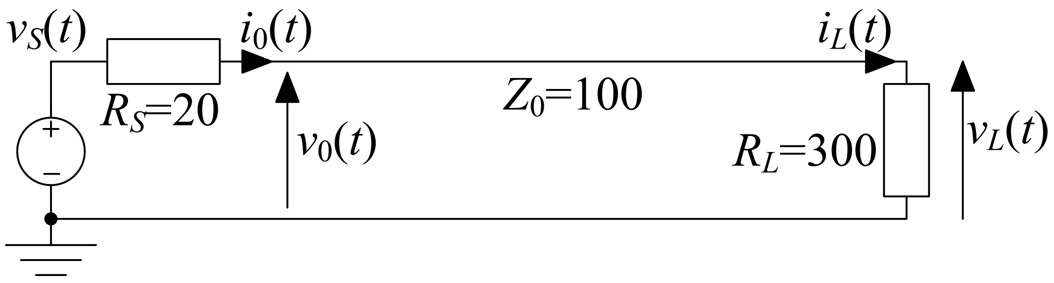
Knowing $f_x(t)$ for $x = x_0$ fixes it for all other x.



Backward wave: $g_x(t)$ is the same but travelling $\leftarrow$: $g_x(t) = g_0\left(t + \frac{x}{u}\right)$.

Voltage and current are: $v_x = f_x + g_x$ and $i_x = \frac{f_x - g_x}{Z_0}$ where i_x is positive in the $+x$ direction ($\rightarrow$) and $Z_0 = \sqrt{\frac{L_0}{C_0}}$

Waveforms of f_x and g_x are determined by the connections at both ends.



$$v_x = f_x + g_x$$
$$i_x = \frac{f_x - g_x}{Z_0}$$

At $x = L$, Ohm's law $\Rightarrow \frac{v_L(t)}{i_L(t)} = R_L \Rightarrow g_L(t) = \frac{R_L - Z_0}{R_L + Z_0} \times f_L(t)$.

Reflection coefficient: $\rho_L = \frac{g_L(t)}{f_L(t)} = \frac{R_L - Z_0}{R_L + Z_0}$

$\rho_L \in [-1, +1]$ and increases with R_L

Knowing $f_x(t)$ for $x = x_0$ now tells you $f_x,\ g_x,\ v_x,\ i_x\ \forall x$

At $x = 0$: $f_0(t) = \frac{Z_0}{R_S + Z_0} v_S(t) + \frac{R_S - Z_0}{R_S + Z_0} g_0(t) = \tau_0 v_S(t) + \rho_0 g_0(t)$

Wave bounces back and forth getting smaller with each reflection:

$$v_S(t) \xrightarrow{\times\tau_0} f_0(t) \xrightarrow{\times\rho_L} g_0(t + \tfrac{2L}{u}) \xrightarrow{\times\rho_0} f_0(t + \tfrac{2L}{u}) \xrightarrow{\times\rho_L} g_0(t + \tfrac{4L}{u}) \xrightarrow{\times\rho_0} \cdots$$

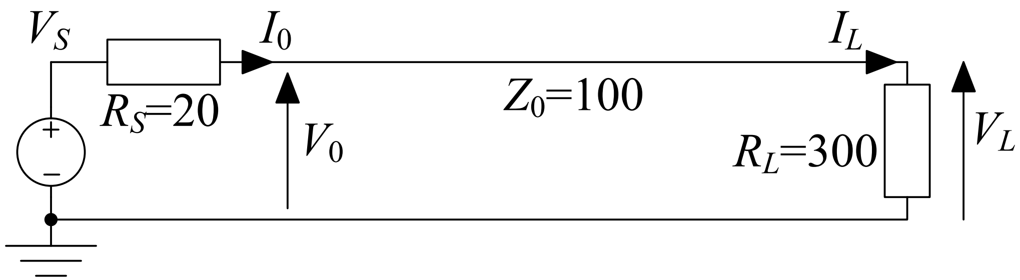
Infinite sum:

$$f_0(t) = \tau_0 v_S(t) + \tau_0 \rho_L \rho_0 v_S(t - \tfrac{2L}{u}) + \ldots = \sum_{i=0}^{\infty} \tau_0 \rho_L^i \rho_0^i v_S\left(t - \tfrac{2Li}{u}\right)$$

Sinewaves are easier because:

1. Use phasors to eliminate t: $f_0(t) = A\cos(\omega t + \phi) \Leftrightarrow F_0 = Ae^{j\phi}$
2. Time delays are just phase shifts:
   $f_x(t) = A\cos\left(\omega\left(t - \frac{x}{u}\right) + \phi\right) \Leftrightarrow F_x = Ae^{j\left(\phi - \frac{\omega}{u}x\right)} = F_0 e^{-jkx}$
   $k = \frac{\omega}{u} = \frac{2\pi}{\lambda}$ is the wavenumber: radians per metre (c.f. ω in rad/s)

As before: $V_x = F_x + G_x$ and $I_x = \frac{F_x - G_x}{Z_0}$



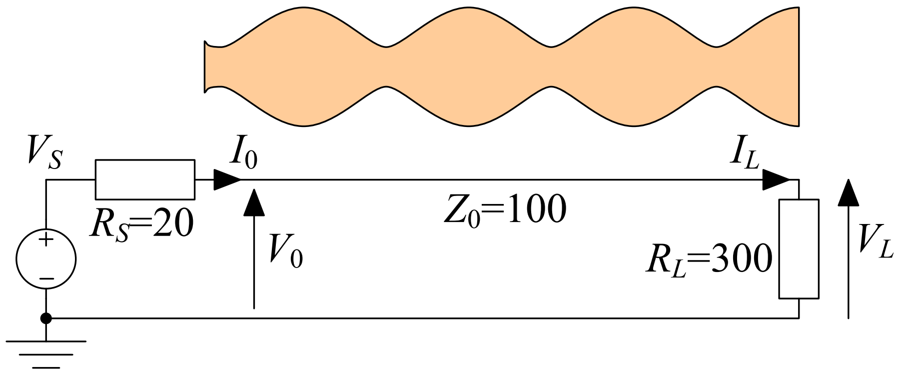
As before:
$G_L = \rho_L F_L$
$F_0 = \tau_0 V_S + \rho_0 G_0$

But $G_0 = F_0 \rho_L e^{-2jkL}$: roundtrip delay of $\frac{2L}{u}$ + reflection at $x = L$.
Substituting for G_0 in source end equation: $F_0 = \tau_0 V_S + \rho_0 F_0 \rho_L e^{-2jkL}$
$\Rightarrow F_0 = \frac{\tau_0}{1 - \rho_0 \rho_L \exp(-2jkL)} V_S$ so no infinite sums needed ☺

Standing Waves

Standing waves arise whenever a wave meets its reflection:
  at positions where the two waves are in phase their amplitudes add
  but where they are anti-phase their amplitudes subtract.

At any point x,
  delay of $\frac{x}{u} \Rightarrow$
  $F_x = F_0 e^{-jkx}$



Backward wave: $G_x = \rho_L F_x e^{-2jk(L-x)}$: reflection + delay of $2\frac{L-x}{u}$
Voltage at x: $V_x = F_x + G_x = F_0 e^{-jkx}\left(1 + \rho_L e^{-2jk(L-x)}\right)$
Voltage Magnitude : $|V_x| = |F_0|\left|1 + \rho_L e^{-2jk(L-x)}\right|$: depends on x

If $\rho_L \geq 0$, max magnitude is $(1 + \rho_L)|F_0|$ whenever $e^{-2jk(L-x)} = +1$
  $\Rightarrow x = L$ or $x = L - \frac{\pi}{k}$ or $x = L - \frac{2\pi}{k}$ or ...

Min magnitude is $(1 - \rho_L)|F_0|$ whenever $e^{-2jk(L-x)} = -1$
  $\Rightarrow x = L - \frac{\pi}{2k}$ or $x = L - \frac{3\pi}{2k}$ or $x = L - \frac{5\pi}{2k}$ or ...