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It is all rather inconsistent ☹.

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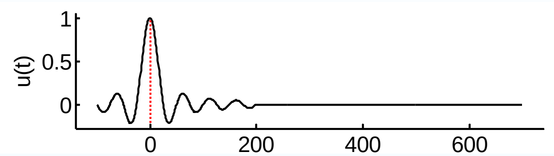

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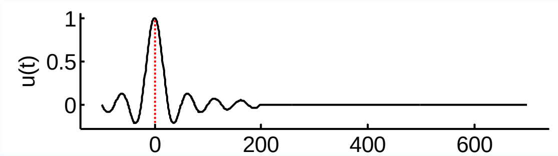

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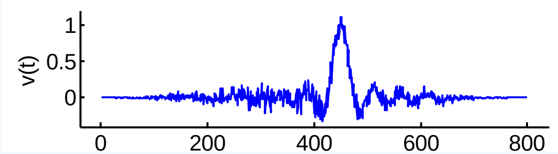
$v(t)$ contains $u(t)$ with an unknown delay and added noise.



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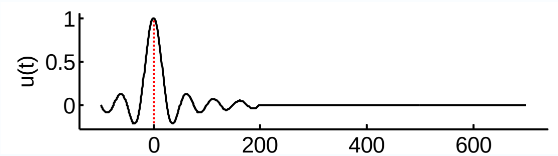

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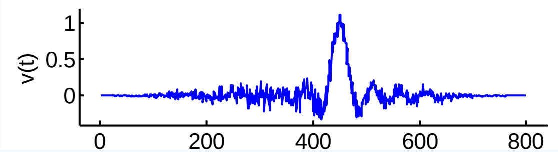

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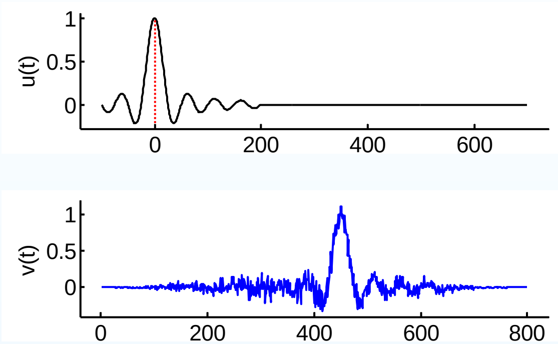

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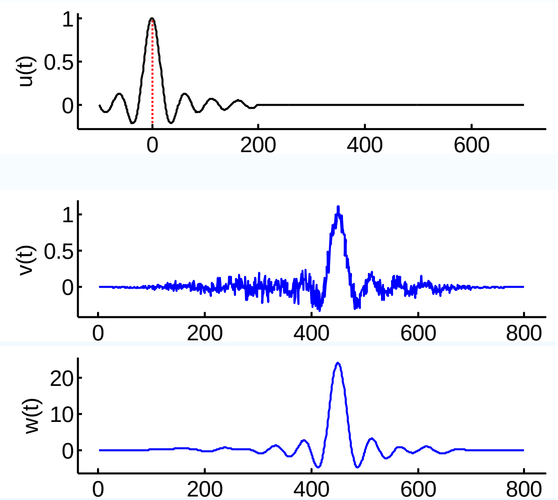

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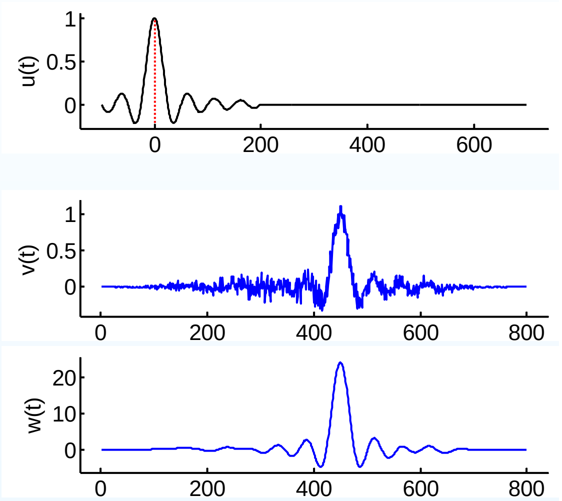

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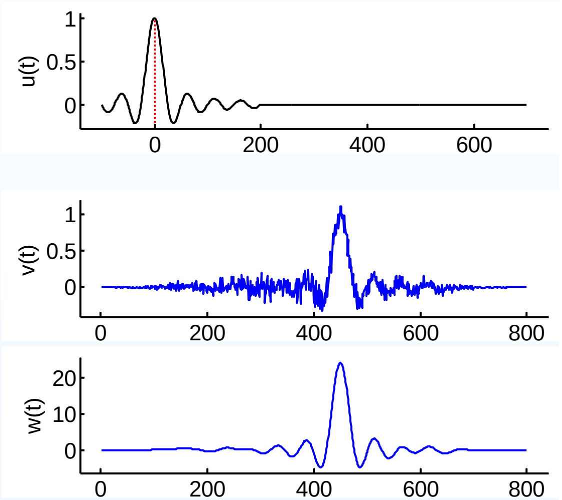

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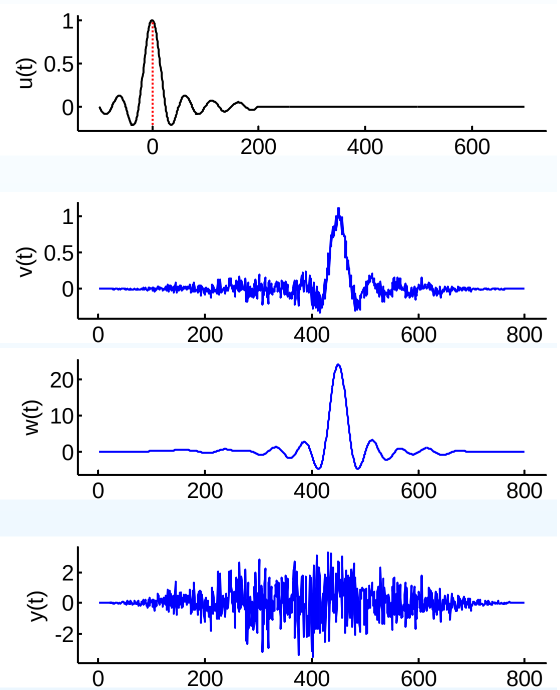

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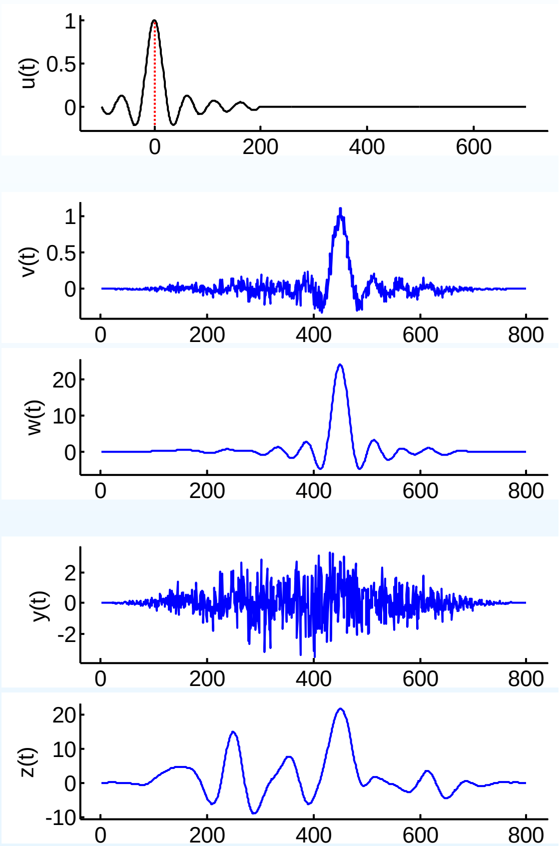

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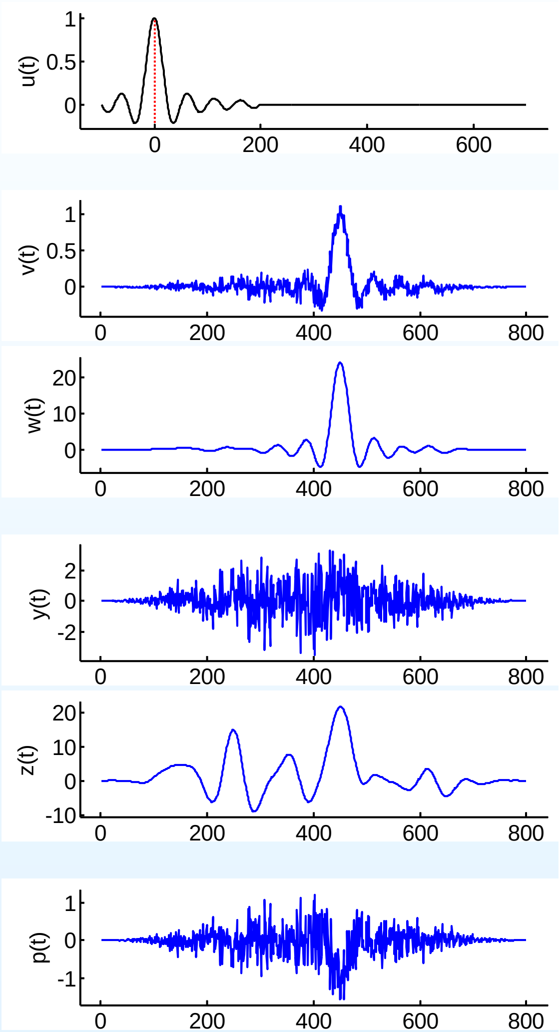

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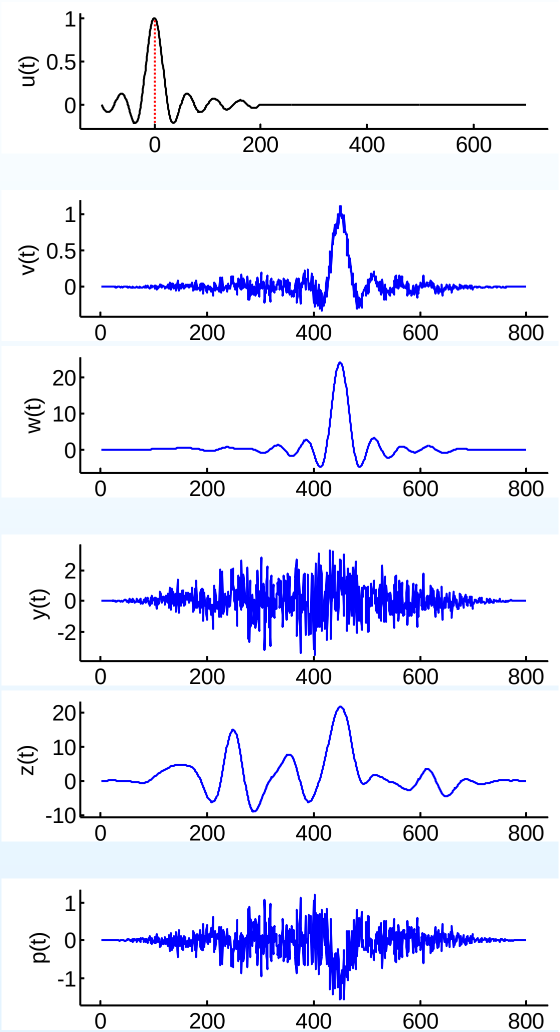

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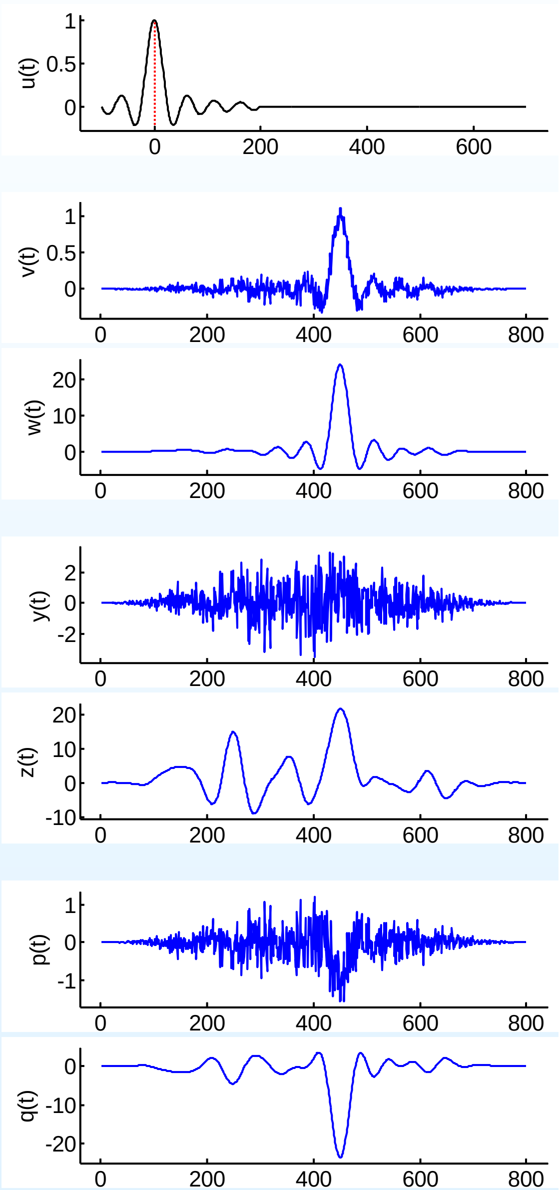

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The Fourier transform of the autocorrelation is the energy spectrum.

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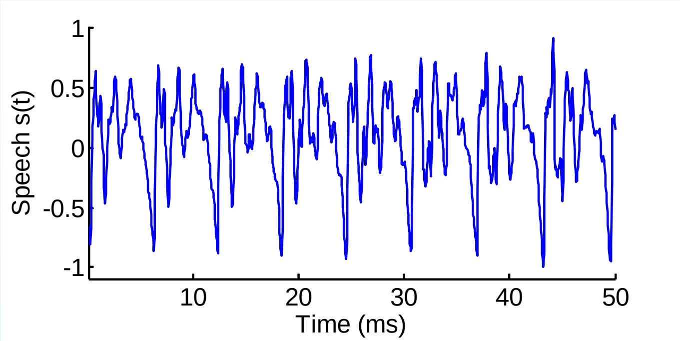

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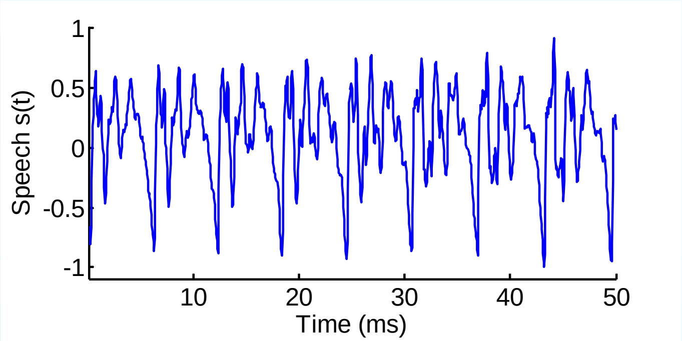

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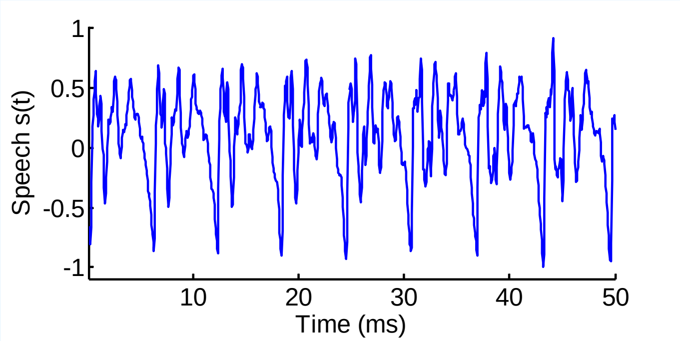

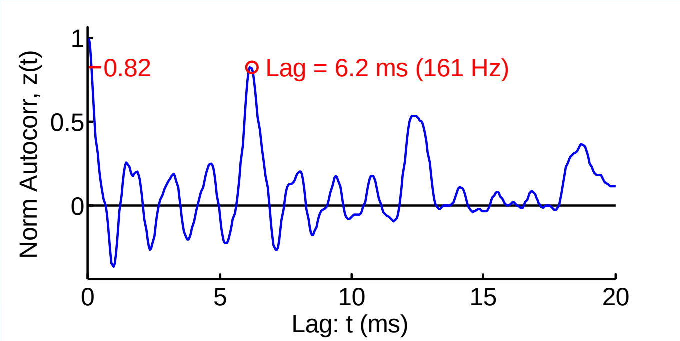

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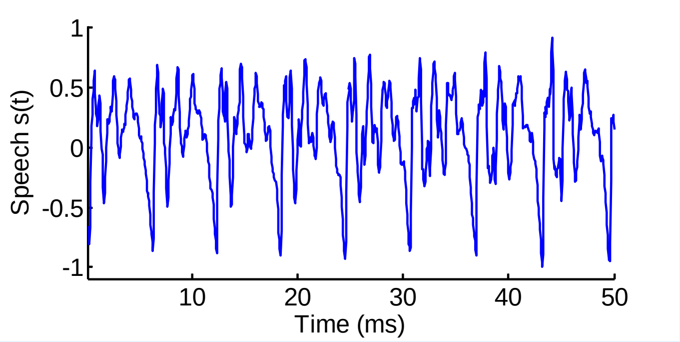

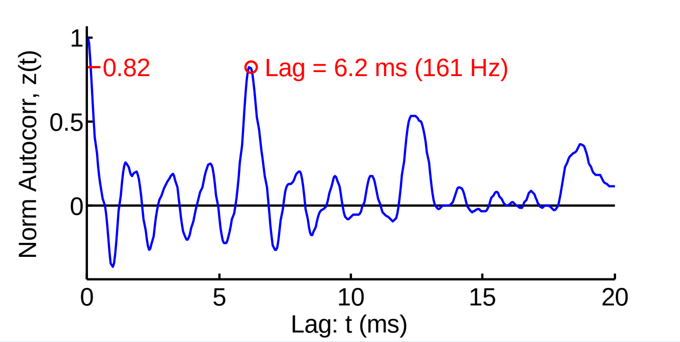

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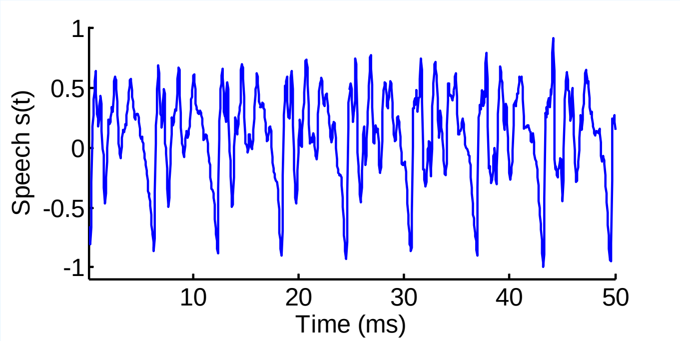
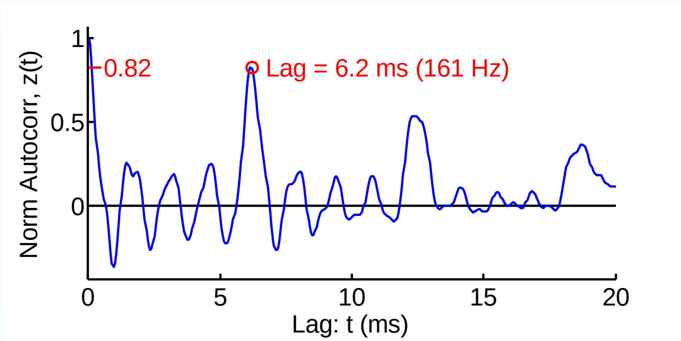




Autocorrelation example

Cross-correlation is used to find when two different signals are similar.
Autocorrelation is used to find when a signal is similar to itself delayed.

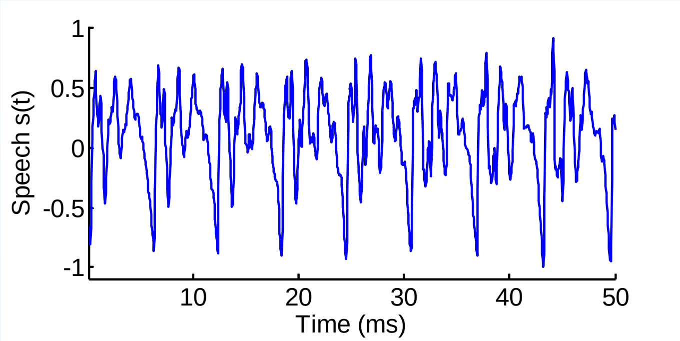
First graph shows $s(t)$ a segment of the microphone signal from the initial vowel of "early" spoken by me. The waveform is "quasi-periodic" = "almost periodic but not quite".

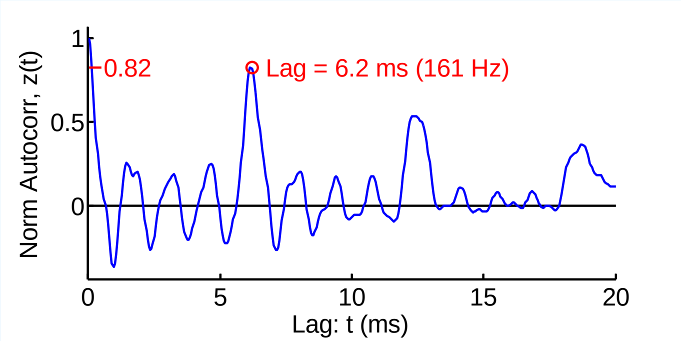
Second graph shows normalized autocorrelation, $z(t) = \frac{s(t) \otimes s(t)}{E_s}$.

$z(0) = 1$ for $t = 0$ since a signal always matches itself exactly.

$z(t) = 0.82$ for $t = 6.2\,\text{ms}$ = one period lag (not an exact match).

$z(t) = 0.53$ for $t = 12.4\,\text{ms}$ = two periods lag (even worse match).





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To obtain formulae for version (3) of the Fourier Transform, $\widehat{U}(\omega)$, substitute into the above formulae: $\widetilde{U}(\omega) = \sqrt{2\pi}\widehat{U}(\omega)$.

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For further details see RHB Chapter 13.1

Spectrogram

Spectrogram of "Merry Christmas" spoken by Mike Brookes (♫)