8: Correlation

- Cross-Correlation
- Signal Matching
- Cross-corr as Convolution
- Normalized Cross-corr
- Autocorrelation
- Autocorrelation example
- Fourier Transform Variants
- Scale Factors
- Summary
- Spectrogram

8: Correlation
The *cross-correlation* between two signals \( u(t) \) and \( v(t) \) is

\[
w(t) = u(t) \otimes v(t) \triangleq \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau
\]

\[
= \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau
\]

[sub: \( \tau \to \tau - t \)]

The complex conjugate, \( u^*(\tau) \) makes no difference if \( u(t) \) is real-valued but makes the definition work even if \( u(t) \) is complex-valued.

**Correlation versus Convolution:**

\[
u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau)v(\tau + t)d\tau \quad \text{[correlation]}
\]

\[
u(t) \ast v(t) = \int_{-\infty}^{\infty} u(\tau)v(t - \tau)d\tau \quad \text{[convolution]}
\]

Unlike convolution, the integration variable, \( \tau \), has the same sign in the arguments of \( u(\cdots) \) and \( v(\cdots) \) so the arguments have a constant difference instead of a constant sum (i.e. \( v(t) \) is not time-flipped).

**Notes:**
(a) The argument of \( w(t) \) is called the “lag” (\( = \) delay of \( u \) versus \( v \)).
(b) Some people write \( u(t) \ast v(t) \) instead of \( u(t) \otimes v(t) \).
(c) Some swap \( u \) and \( v \) and/or negate \( t \) in the integral.

It is all rather inconsistent 😞.
Cross correlation is used to find where two signals match: $u(t)$ is the test waveform.

**Example 1:**

$v(t)$ contains $u(t)$ with an unknown delay and added noise.

$$w(t) = u(t) \otimes v(t) = \int u^*(\tau - t)v(\tau)d\tau$$

gives a peak at the time lag where $u(\tau - t)$ best matches $v(\tau)$; in this case at $t = 450$

**Example 2:**

$y(t)$ is the same as $v(t)$ with more noise

$z(t) = u(t) \otimes y(t)$ can still detect the correct time delay (hard for humans)

**Example 3:**

$p(t)$ contains $-u(t)$ so that

$q(t) = u(t) \otimes p(t)$ has a negative peak
Cross-correlation as Convolution

Correlation: \( w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)\,d\tau \)

If we define \( x(t) = u^*(-t) \) then
\[
x(t) \ast v(t) \triangleq \int_{-\infty}^{\infty} x(t - \tau)v(\tau)\,d\tau = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)\,d\tau = u(t) \otimes v(t)
\]

Fourier Transform of \( x(t) \):
\[
X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}\,dt = \int_{-\infty}^{\infty} u^*(-t)e^{-i2\pi ft}\,dt = \left( \int_{-\infty}^{\infty} u(t)e^{-i2\pi ft}\,dt \right)^* = U^*(f)
\]
So \( w(t) = x(t) \ast v(t) \iff W(f) = X(f)V(f) = U^*(f)V(f) \)

Hence the Cross-correlation theorem:
\[
w(t) = u(t) \otimes v(t) \iff W(f) = U^*(f)V(f) = u^*(-t) \ast v(t)
\]

Note that, unlike convolution, correlation is not associative or commutative:
\[
v(t) \otimes u(t) = v^*(-t) \ast u(t) = u(t) \ast v^*(-t) = w^*(-t)
\]
Normalized Cross-correlation

Correlation: \( w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau) d\tau \)

If we define \( y(t) = u(t - t_0) \) for some fixed \( t_0 \), then \( E_y = E_u \):

\[
E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |u(t - t_0)|^2 dt
= \int_{-\infty}^{\infty} |u(\tau)|^2 d\tau = E_u \quad \text{[} t \to \tau + t_0 \text{]}
\]

Cauchy-Schwarz inequality: \( \left| \int_{-\infty}^{\infty} y^*(\tau)v(\tau) d\tau \right|^2 \leq E_y E_v \)

\[
\Rightarrow |w(t_0)|^2 = \left| \int_{-\infty}^{\infty} u^*(\tau - t_0)v(\tau) d\tau \right|^2 \leq E_y E_v = E_u E_v
\]

but \( t_0 \) was arbitrary, so we must have \( |w(t)| \leq \sqrt{E_u E_v} \) for all \( t \)

We can define the **normalized cross-correlation**

\[
z(t) = \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}}
\]

with properties: (1) \( |z(t)| \leq 1 \) for all \( t \)

(2) \( |z(t_0)| = 1 \Leftrightarrow v(\tau) = \alpha u(\tau - t_0) \) with \( \alpha \) constant
[Cauchy-Schwarz Inequality Proof]

We want to prove the Cauchy-Schwarz Inequality:

\[ \left| \int_{-\infty}^{\infty} u^*(t)v(t) dt \right|^2 \leq E_u E_v \]

where \( E_u \triangleq \int_{-\infty}^{\infty} |u(t)|^2 dt \).

You do not need to memorize this proof

Suppose we define \( w \triangleq \int_{-\infty}^{\infty} u^*(t)v(t) dt \). Then,

\[
0 \leq \int \left| E_v u(t) - w^*v(t) \right|^2 dt \\
= \int (E_v u^*(t) - wv^*(t))(E_v u(t) - w^*v(t)) dt \\
= E_v^2 \int u^*(t)u(t) dt + |w|^2 \int v^*(t)v(t) dt - w^*E_v \int u^*(t)v(t) dt - wE_v \int u(t)v^*(t) dt \\
= E_v^2 \int |u(t)|^2 dt + |w|^2 \int |v(t)|^2 dt - E_vw^*w - E_vww^* \]  

\[
= E_v^2 E_u + |w|^2 E_v - 2 |w|^2 E_v = E_v \left( E_u E_v - |w|^2 \right) \]

Unless \( E_v = 0 \) (in which case, \( v(t) \equiv 0 \) and the C-S inequality is true), we must have \( |w|^2 \leq E_u E_v \) which proves the C-S inequality.

Also, \( E_u E_v = |w|^2 \) only if we have equality in the first line,

that is, \( \int \left| E_v u(t) - w^*v(t) \right|^2 dt = 0 \) which implies that the integrand is zero for all \( t \).

This implies that \( u(t) = \frac{w^*}{E_v} v(t) \).

So we have shown that \( E_u E_v = |w|^2 \) if and only if \( u(t) \) and \( v(t) \) are proportional to each other.
The correlation of a signal with itself is its autocorrelation:
\[ w(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \]

The autocorrelation at zero lag:
\[ w(0) = \int_{-\infty}^{\infty} u^*(\tau - 0)u(\tau)d\tau \]
\[ = \int_{-\infty}^{\infty} u^*(\tau)u(\tau)d\tau \]
\[ = \int_{-\infty}^{\infty} |u(\tau)|^2 d\tau = E_u \]

The autocorrelation at zero lag, \( w(0) \), is the energy of the signal.

The normalized autocorrelation:
\[ z(t) = \frac{u(t) \otimes u(t)}{E_u} \]
satisfies \( z(0) = 1 \) and \( |z(t)| \leq 1 \) for any \( t \).

Wiener-Khinchin Theorem: [Cross-correlation theorem when \( v(t) = u(t) \)]
\[ w(t) = u(t) \otimes u(t) \iff W(f) = U^*(f)U(f) = |U(f)|^2 \]

The Fourier transform of the autocorrelation is the energy spectrum.
Cross-correlation is used to find when two different signals are similar. Autocorrelation is used to find when a signal is similar to itself delayed.

First graph shows $s(t)$ a segment of the microphone signal from the initial vowel of “early” spoken by me. The waveform is “quasi-periodic” = “almost periodic but not quite”.

Second graph shows normalized autocorrelation, $z(t) = \frac{s(t) \otimes s(t)}{E_s}$.

$z(0) = 1$ for $t = 0$ since a signal always matches itself exactly.

$z(t) = 0.82$ for $t = 6.2 \text{ ms} = \text{one period lag (not an exact match)}$.

$z(t) = 0.53$ for $t = 12.4 \text{ ms} = \text{two periods lag (even worse match)}$. 
There are three different versions of the Fourier Transform in current use.

(1) Frequency version (we have used this in lectures)

\[ U(f) = \int_{-\infty}^{\infty} u(t) e^{-i2\pi ft} \, dt \quad u(t) = \int_{-\infty}^{\infty} U(f) e^{i2\pi ft} \, df \]

- Used in the communications/broadcasting industry and textbooks.
- The formulae do not need scale factors of \(2\pi\) anywhere.

(2) Angular frequency version

\[ \tilde{U}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} \, dt \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(\omega) e^{i\omega t} \, d\omega \]

Continuous spectra are unchanged: \( \tilde{U}(\omega) = U(f) = U(\frac{\omega}{2\pi}) \)

However \(\delta\)-function spectral components are multiplied by \(2\pi\) so that

\[ U(f) = \delta(f - f_0) \quad \Rightarrow \quad \tilde{U}(\omega) = 2\pi \times \delta(\omega - 2\pi f_0) \]

- Used in most signal processing and control theory textbooks.

(3) Angular frequency + symmetrical scale factor

\[ \hat{U}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t) e^{-i\omega t} \, dt \quad u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{U}(\omega) e^{i\omega t} \, d\omega \]

In all cases \( \hat{U}(\omega) = \frac{1}{\sqrt{2\pi}} \tilde{U}(\omega) \)

- Used in many Maths textbooks (mathematicians like symmetry)
Fourier Transform using Angular Frequency:

\[
\tilde{U}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{U}(\omega) e^{i\omega t} d\omega
\]

Any formula involving \(\int df\) will change to \(\frac{1}{2\pi} \int d\omega\) \(\text{[since } d\omega = 2\pi df\}\)

Parseval’s Theorem:

\[
\int u^*(t)v(t)dt = \frac{1}{2\pi} \int \tilde{U}^*(\omega)\tilde{V}(\omega) d\omega
\]

\[
E_u = \int |u(t)|^2 dt = \frac{1}{2\pi} \int \left|\tilde{U}(\omega)\right|^2 d\omega
\]

Waveform Multiplication: (convolution implicitly involves integration)

\[w(t) = u(t)v(t) \Rightarrow \tilde{W}(\omega) = \frac{1}{2\pi} \tilde{U}(\omega) * \tilde{V}(\omega)\]

Spectrum Multiplication: (multiplication \(\neq\) integration)

\[w(t) = u(t) * v(t) \Rightarrow \tilde{W}(\omega) = \tilde{U}(\omega)\tilde{V}(\omega)\]

To obtain formulae for version (3) of the Fourier Transform, \(\hat{U}(\omega)\), substitute into the above formulae: \(\tilde{U}(\omega) = \sqrt{2\pi}\hat{U}(\omega)\).
• **Cross-Correlation**: $w(t) = u(t) \otimes v(t) = \int_{-\infty}^{\infty} u^*(\tau - t)v(\tau)d\tau$
  ○ Used to find similarities between $v(t)$ and a delayed $u(t)$
  ○ Cross-correlation theorem: $W(f) = U^*(f)V(f)$
  ○ Cauchy-Schwarz Inequality: $|u(t) \otimes v(t)| \leq \sqrt{E_u E_v}$
    ▶ Normalized cross-correlation: $\left| \frac{u(t) \otimes v(t)}{\sqrt{E_u E_v}} \right| \leq 1$

• **Autocorrelation**: $x(t) = u(t) \otimes u(t) = \int_{-\infty}^{\infty} u^*(\tau - t)u(\tau)d\tau \leq E_u$
  ○ Wiener-Khinchin: $X(f) = \text{energy spectral density}$, $|U(f)|^2$
  ○ Used to find periodicity in $u(t)$

• **Fourier Transform using $\omega$**:
  ○ Continuous spectra unchanged; spectral impulses multiplied by $2\pi$
  ○ In formulae: $\int df \rightarrow \frac{1}{2\pi} \int d\omega$; $\omega$-convolution involves an integral

For further details see RHB Chapter 13.1
Spectrogram of “Merry Christmas” spoken by Mike Brookes
All waveforms have period $T = 1$. $\delta_{\text{condition}}$ is 1 whenever “condition” is true and otherwise 0.

| Waveform         | $x(t)$ for $|t| < 0.5$                          | $X_n$                           |
|------------------|-----------------------------------------------|---------------------------------|
| Square wave      | $2\delta_{|t|<0.25} - 1$                      | $\frac{2\sin 0.5\pi n}{\pi n} \times \delta_{n \neq 0}$ |
| Pulse of width $d$ | $\delta_{|t|<0.5d}$                       | $\frac{\sin \pi dn}{\pi n}$     |
| Sawtooth wave    | $2t$                                         | $\frac{i(-1)^n}{\pi n} \times \delta_{n \neq 0}$ |
| Triangle wave    | $1 - 4|t|$                                    | $\frac{2(1-(-1)^n)}{\pi^2 n^2}$  |
You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$

<table>
<thead>
<tr>
<th>Property</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Forward</strong></td>
<td>$x(t)$</td>
<td>$\int x(t)e^{-i2\pi ft} dt$</td>
</tr>
<tr>
<td><strong>Inverse</strong></td>
<td>$\int X(f)e^{i2\pi ft} df$</td>
<td>$X(f)$</td>
</tr>
<tr>
<td><strong>Spectral Zero</strong></td>
<td>$\int x(t)dt$</td>
<td>$= X(0)$</td>
</tr>
<tr>
<td><strong>Temporal Zero</strong></td>
<td>$x(0)$</td>
<td>$= \int X(f)df$</td>
</tr>
<tr>
<td><strong>Duality</strong></td>
<td>$X(t)$</td>
<td>$x(-f)$</td>
</tr>
<tr>
<td><strong>Reversal</strong></td>
<td>$x(-t)$</td>
<td>$X(-f)$</td>
</tr>
<tr>
<td><strong>conjugate</strong></td>
<td>$x^*(t)$</td>
<td>$X^*(-f)$</td>
</tr>
<tr>
<td><strong>Temporal Derivative</strong></td>
<td>$\frac{d^n}{dt^n} x(t)$</td>
<td>$(i2\pi f)^n X(f)$</td>
</tr>
<tr>
<td><strong>Spectral Derivative</strong></td>
<td>$(-i2\pi t)^n x(t)$</td>
<td>$\frac{d^n}{df^n} X(f)$</td>
</tr>
<tr>
<td><strong>Integral</strong></td>
<td>$\int_{-\infty}^{t} x(\tau)d\tau$</td>
<td>$\frac{1}{i2\pi f} X(f) + \frac{1}{2}X(0)\delta(f)$</td>
</tr>
<tr>
<td><strong>Scaling</strong></td>
<td>$x(\alpha t + \beta)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td><strong>Time Shift</strong></td>
<td>$x(t - T)$</td>
<td>$X(f)e^{-i2\pi ft}$</td>
</tr>
<tr>
<td><strong>Frequency Shift</strong></td>
<td>$x(t)e^{i2\pi Ft}$</td>
<td>$X(f - F)$</td>
</tr>
</tbody>
</table>
You need not memorize these properties. All integrals are $\int_{-\infty}^{\infty}$.

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<tr>
<th>Property</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$\alpha x(t) + \beta y(t)$</td>
<td>$\alpha X(f) + \beta Y(f)$</td>
</tr>
<tr>
<td>Multiplication</td>
<td>$x(t)y(t)$</td>
<td>$X(f) * Y(f)$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$x(t) * y(t)$</td>
<td>$X(f)Y(f)$</td>
</tr>
<tr>
<td>Correlation</td>
<td>$x(t) \otimes y(t)$</td>
<td>$X^*(f)Y(f)$</td>
</tr>
<tr>
<td>Autocorrelation</td>
<td>$x(t) \otimes x(t)$</td>
<td>$</td>
</tr>
<tr>
<td>Parseval or Plancherel</td>
<td>$\int x^*(t)y(t)dt$</td>
<td>$\int X^*(f)Y(f)df$</td>
</tr>
<tr>
<td>Repetition</td>
<td>$\sum_n x(t-nT)$</td>
<td>$\left</td>
</tr>
<tr>
<td>Sampling</td>
<td>$\sum_n x(nT)\delta(t-nT)$</td>
<td>$\left</td>
</tr>
<tr>
<td>Modulation</td>
<td>$x(t)\cos(2\pi Ft)$</td>
<td>$\frac{1}{2}X(f-F) + \frac{1}{2}X(f+F)$</td>
</tr>
</tbody>
</table>

Convolution: $x(t) * y(t) = \int x(\tau)y(t-\tau)d\tau$

Cross-correlation: $x(t) \otimes y(t) = \int x^*(\tau)y(\tau+t)d\tau = \int x^*(\tau-t)y(\tau)d\tau$
You need not memorize these pairs.

<table>
<thead>
<tr>
<th>$x(t)$</th>
<th>$X(f)$</th>
<th>$x(t)$</th>
<th>$X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
<td>1</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$\text{rect}(t)$</td>
<td>$\frac{\sin(\pi f)}{\pi f}$</td>
<td>$\frac{\sin(t)}{t}$</td>
<td>$\pi \text{rect}(\pi f)$</td>
</tr>
<tr>
<td>$\text{tri}(t)$</td>
<td>$\frac{\sin^2(\pi f)}{\pi^2 f^2}$</td>
<td>$\frac{\sin^2(t)}{t^2}$</td>
<td>$\pi \text{tri}(\pi f)$</td>
</tr>
<tr>
<td>$\cos(2\pi \alpha t)$</td>
<td>$\frac{1}{2} \delta(f + \alpha) + \frac{1}{2} \delta(f - \alpha)$</td>
<td>$\sin(2\pi \alpha t)$</td>
<td>$\frac{i}{2} \delta(f + \alpha) - \frac{i}{2} \delta(f - \alpha)$</td>
</tr>
<tr>
<td>$e^{-\alpha t} u(t)$</td>
<td>$\frac{1}{\alpha + 2\pi i f}$</td>
<td>$te^{-\alpha t} u(t)$</td>
<td>$\frac{1}{(\alpha + 2\pi i f)^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha</td>
<td>t</td>
<td>}$</td>
<td>$\frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$</td>
</tr>
<tr>
<td>$\text{sgn}(t)$</td>
<td>$\frac{1}{i \pi f}$</td>
<td>$u(t)$</td>
<td>$\frac{1}{2} \delta(f) + \frac{1}{2\pi i f}$</td>
</tr>
<tr>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT)$</td>
<td>$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\frac{f - k}{T}\right)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Elementary Functions:

- $\text{rect}(t) = \begin{cases} 1, & |t| < 0.5 \\ 0, & \text{elsewhere} \end{cases}$
- $\text{tri}(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{elsewhere} \end{cases}$
- $\text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$
- $u(t) = \frac{1}{2} (1 + \text{sgn}(t)) = \begin{cases} 0, & x < 0 \\ 0.5, & x = 0 \\ 1, & x > 0 \end{cases}$