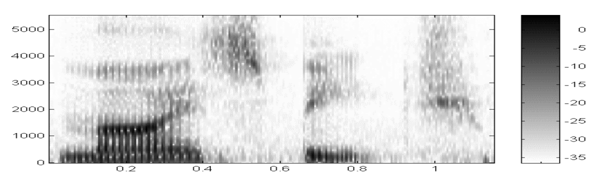
Lecture 3

Time-Frequency Representation

This lecture is concerned with the spectrogram as a representation of how the frequency components within a signal vary with time.

– Define what we mean by normalised time and frequency
– Define the short-term discrete fourier transform
– Look at the effect of different window lengths on time and frequency resolution
– Derive a quantitative description of the frequency resolution of the short term DFT and compare the performance of common windows
– Derive a quantitative description of the time resolution of the short term DFT
– Explain the uncertainty principle and illustrate its effects with some examples
– Review some of the properties of the DFT.



Normalised Time

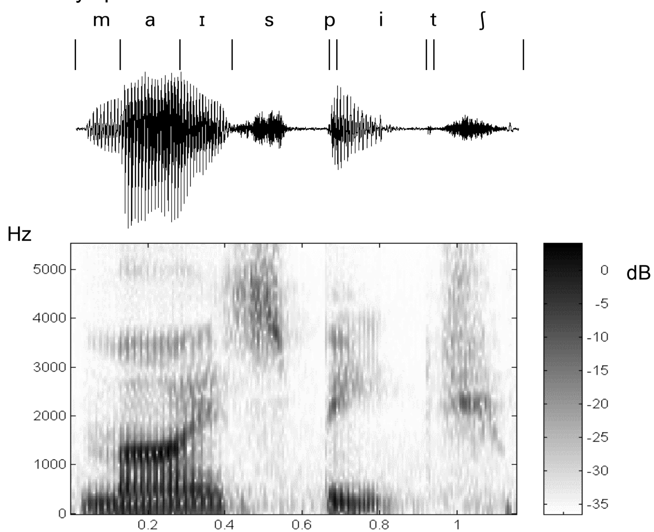
With sampled data systems it is customary to change the time scale and work in units of the sample period.

– In normalised units the sampling frequency (f_s) and period (T) both equal 1. They can therefore be omitted from equations: any such omissions can be deduced using dimensional consistency arguments.
– The nyquist frequency is ½ Hz = π radians/second
– To convert back to real units:
  - any time quantity must be multiplied by the real sample period (or divided by the real sample frequency)
  - any frequency or angular frequency quantity must be multiplied by the real sample frequency (or divided by the real sample period)

Spectrogram

The spectrogram shows the energy in a signal at each frequency and at each time. We calculate this by evaluating the short-term discrete fourier transform.

"my speech"



– Dark areas of spectrogram show high intensity

Short-Term Discrete Fourier Transform

We often want to estimate the "power spectrum" of a non-stationary signal at a particular instant of time.

Multiply by a finite length window and take the DFT.

For a window of length N ending on sample m we have:

$$X(k;m) = \sum_{i=0}^{N-1} w(i)x(m-i)\exp\left(-\frac{2\pi j}{N}k(m-i)\right)$$

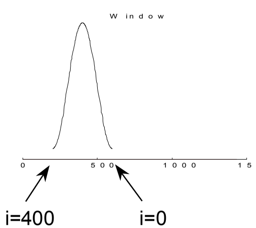
[k=frequency, m=time]

Notes:

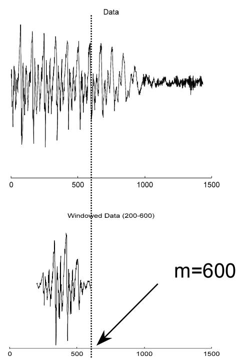
- $|X(k;m)|^2$ gives the power at a frequency of k/N Hz (normalized) for a window centred at $m-½(N-1)$.
- the $(m-i)$ term in the exponent means that the phase origin remains consistent by cancelling out the linear phase shift introduced by a delay of m samples.
- the window samples are numbered backwards in time (for convenience later) hence the summation is performed backwards in time.
- The values $X(k;m)$ are based on the N signal values from $m–N+1$ to m.
- the frequency resolution is $1/N$ Hz (normalised units)
- the spectrum is periodic and (since w and x are real) conjugate symmetric:
  $$X(k) = X(k+N) = X^*(-k)$$
- there are only $½N+1$ independent frequency values

$$X(k;m)=\sum_{i=0}^{N-1} w(i)x(m-i)\exp\left(-\frac{2\pi j}{N}k(m-i)\right)$$

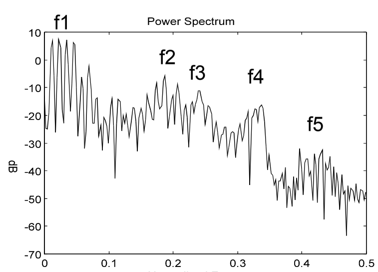
Hamming window of length 401 samples centred on 400.



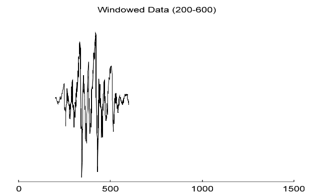
/ɪs/ from "my speech"



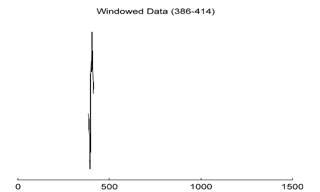
Line spectrum from larynx oscillation (at about 0.01 normalised Hz) is superimposed on vocal tract resonances.



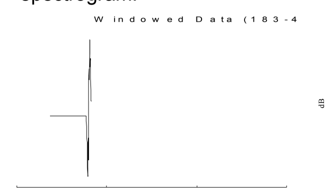
Three different Hamming windows: (N=401)



Short window eliminates the fine detail (N=29):



Zero padded window gives more spectrum points and an illusion of more detail (N=232=29×16). This is the normal case for a spectrogram.



$$X(k;m) = \sum_{i=0}^{N-1} w(i)x(m-i)\exp\left(-\frac{2\pi j}{N}k(m-i)\right)$$

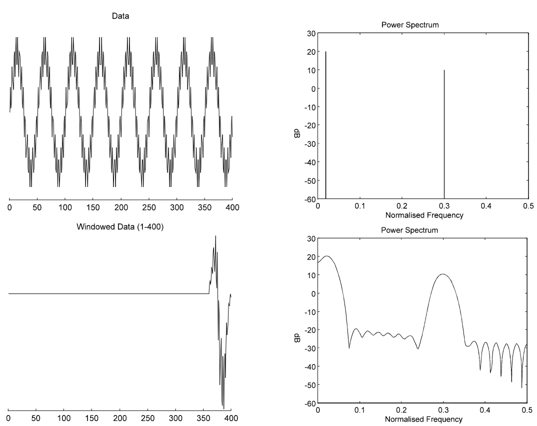
By setting $r = N - 1 - i$ we can rewrite this as:

$$X(k;m) = \exp\left(-\frac{2\pi j(m-N+1)}{N}k\right) \times \sum_{r=0}^{N-1} y_m(r)\exp\left(-\frac{2\pi j}{N}kr\right)$$

where $y_m(r) = w(N-1-r)x(m-N+1+r)$

This is a standard DFT multiplied by a phase-shift term that is proportional to k: this compensates for the starting time of the window: $m–N+1$

$y(r)$ is a *product* of two signals so its DFT is the *convolution* of the DFT's of $w(N–1–r)$ and $x(m–N+1+r)$



The –6dB and –∞ normalised bandwidth for an N-point window = a/N and b/N respectively. Common windows & values for a & b are shown below. In the formulae $c_k = \cos(2k\pi(n - ½N)/N)$

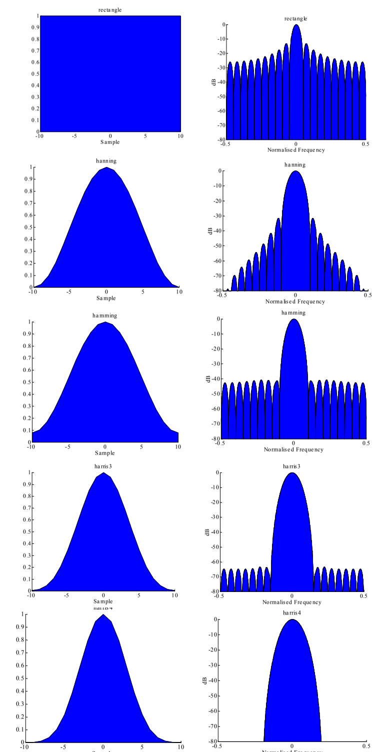
Rectangular: $w(n) \equiv 1$
a=1.21, b=2
Sidelobe = -13dB

Hanning: $0.5 + 0.5c_1$
a=1.65, b=4
Sidelobe = –23dB

Hamming: $0.54 + 0.46c_1$
a=1.81, b=4
Sidelobe = –43dB

Blackman-Harris: 3 term
$0.423 + 0.498c_1 + 0.079c_2$
a=1.81, b=6
Sidelobe = –67dB

Blackman-Harris: 4 term
$0.359 + 0.488c_1 + 0.141c_2 + 0.012c_3$
a=2.72, b=8
Sidelobe = –92dB



Time Resolution: Filter-Bank Viewpoint

Concentrate on one particular value of k and define:

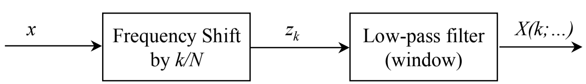
$$z_k(r) = x(r)\exp\left(-\frac{2\pi j}{N}kr\right)$$

$z_k(r)$ is just the same as $x(r)$ but shifted down in frequency by k/N. E.g. if x is a complex exponential at frequency f

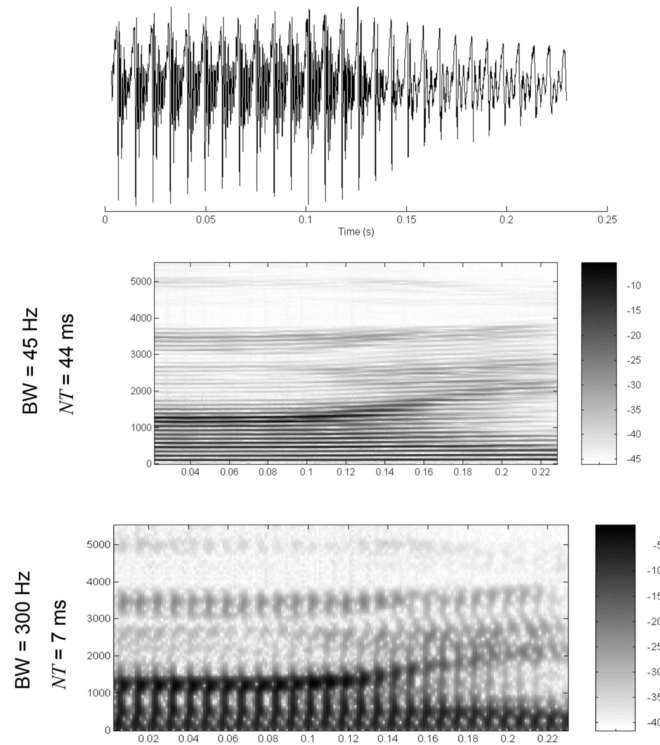
$$x(r) = \exp(2\pi jfr) \quad \Rightarrow \quad z_k(r) = \exp(2\pi j(f - k/N)r)$$

Now we have: $$X(k;m) = \sum_{i=0}^{N-1} w(i)x(m-i)\exp\left(-\frac{2\pi j}{N}k(m-i)\right)$$
$$= \sum_{i=0}^{N-1} w(i)z_k(m-i)$$

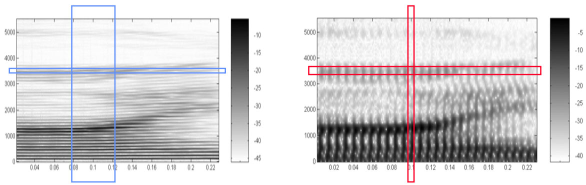
Thus the k'th frequency bin is a filtered version of z_k in which the filter has an impulse response of $w(i)$. From the previous slide, this is a low-pass filter with a –6dB bandwidth of ½a/N.



/aɪ/ from "my" with 45Hz and 300Hz bandwidth spectrograms



/aɪ/ from "my" with 45Hz and 300Hz bandwidth spectrograms



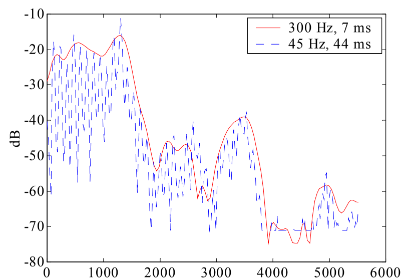
45 Hz, 44 ms

300 Hz, 7 ms

Vertical slice through spectrogram:

mT = 0.1 s

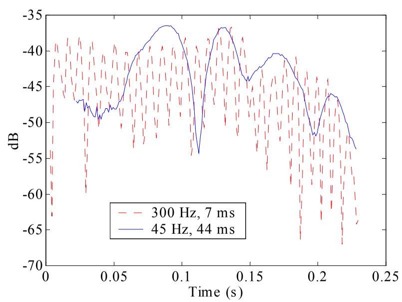
45 Hz gives finer frequency resolution



Horizontal slice through spectrogram:

k/NT = 3.5 kHz

300 Hz gives finer time resolution



Uncertainty Principle

You cannot get good time resolution and good frequency resolution from the same spectrogram.

– Duration of window = N/f_s

– Frequency resolution = $f_s \times a/N$
  - Equal amplitude frequency components with this separation will give distinct peaks
  - Most windows have $a \approx 2$ (see earlier)

– Time resolution = $2N/af_s$
  - Amplitude variations with this period will be attenuated by 6 dB.

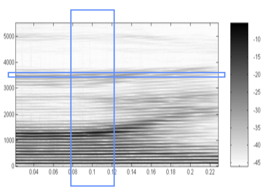
For all window functions, the product of the time and frequency resolutions is equal to 2.

Linguistic analysis typically uses a window length of 10–20 ms. The transfer function of the vocal tract does not change significantly in this time.

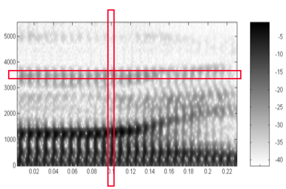
Overlapping Windows

To keep all the information about time variation of spectral components, you need only sample the spectrum twice as fast as the spectral magnitudes are varying. Using normalised frequencies:

- –∞ dB bandwidth for an N-point window = b/N
  - $b = 4$ for a Hamming window
- Significant variation of spectral components occurs at frequencies below $\frac{1}{2}b/N$
- we must sample the spectrum at a frequency of b/N
  - the separation between spectral samples is the window width divided by b

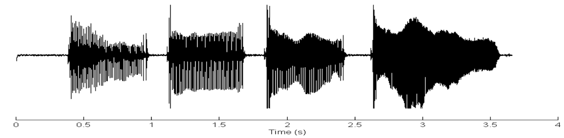


45 Hz, 44 ms

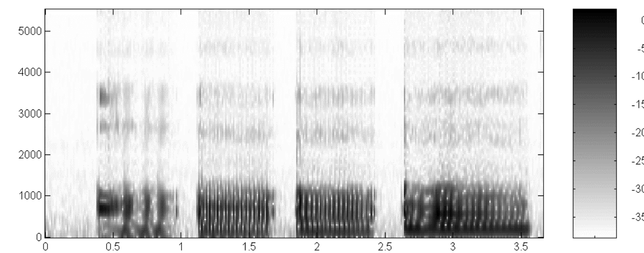


300 Hz, 7 ms

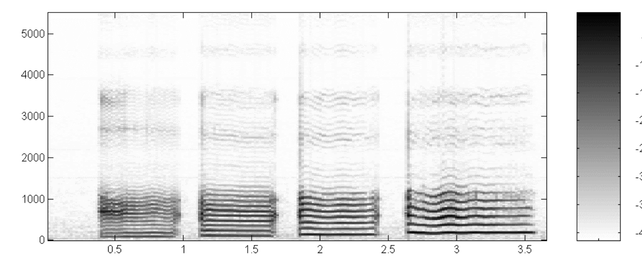
/ɑ/ (“ah”) sung as an arpedgio



300 Hz Bandwidth: Note constant formant frequencies



50 Hz Bandwidth: Note harmonic spacing increases + f_x warbles



$f_x = f_0$ $1.25f_0$ $1.5f_0$ $2f_0$

DFT Properties

$$X_k = \sum_{m=0}^{N-1} x_m \exp\left(-\frac{2\pi j}{N} km\right) = X(z) \text{ evaluated at } z = \exp\left(2\pi j \frac{k}{N}\right)$$

- Exact line-spectrum of a periodic signal $\{x_m\}$
- Sampled continuous spectrum of zero-extended $\{x_m\}$
- Sampled continuous spectrum of infinite $\{x_m\}$ convolved with spectrum of rectangular window
- FFT is an *algorithm* for calculating DFT in time $\propto N \log N$

Energy Conservation (Parseval's theorem)

$$E_x = \sum_{m=0}^{N-1} x_m^2 = \frac{1}{N} \sum_{k=0}^{N-1} X_k^2$$

Symmetries $\{x_m\} \Rightarrow \{X_k\}$

Discrete $\Rightarrow$ Periodic: $X_{k+N} = X_k$
Real $\Rightarrow$ Hermitian: $X_{-k} = X^*_k$
Periodic: $x_{m+N/r} = x_m$ $\Rightarrow$ Discrete: $X_k = 0$ for $k \neq ir$
Skew Periodic: $x_{m+N/2r} = -x_m$ $\Rightarrow$ Odd Harmonics: $X_k = 0$ for $k \neq (2i+1)r$
Even: $x_m = x_{N-m}$ $\Rightarrow$ Real
Odd: $x_m = -x_{N-m}$ $\Rightarrow$ Purely Imaginary
Real & Even $\Rightarrow$ Real & Even
Real & Odd $\Rightarrow$ Purely Imaginary and Odd