Lecture 4

## Facts about Filters

This lecture reviews some well known facts about filters and introduces some less known ones that will be needed later on.

- Derive the power response of first order FIR and IIR filters and relate this to the geometry of the pole-zero diagram.
- Relate the bandwidth of a 2nd-order resonance to the geometry of the pole-zero diagram.
- Describe the bandwidth expansion transformation of a filter.
- Describe the effect of reversing the coefficients of a filter.
- Derive expressions for the log frequency response and its average value.



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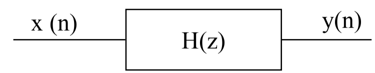
## 1st order FIR filter




$$H(z) = 1 - az^{-1} \quad \Leftrightarrow \quad y(n) = x(n) - ax(n-1)$$

Filter has a single zero at $z = a = re^{j\theta}$.

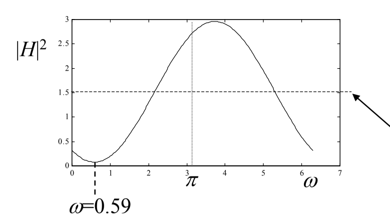
Frequency response of filter is given by: $H(e^{j\omega}) = 1 - ae^{-j\omega}$

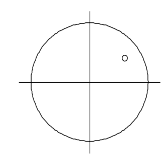
Power response of filter is given by:

$$\begin{aligned}|H(e^{j\omega})|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= \left(1 - ae^{-j\omega}\right)\left(1 - a^*e^{+j\omega}\right) \\ &= 1 + r^2 - 2r\cos(\omega - \theta)\end{aligned}$$

Example: $a = 0.6 + 0.4j = 0.72e^{0.59j}$

$$|H(e^{j\omega})|^2 = 1.52 - 1.44\cos(\omega - 0.59)$$



## Log Frequency Response for |a|<1

We can calculate the log response of the filter

$$\log\left(H(e^{j\omega})\right) = \log\left(1 - ae^{-j\omega}\right)$$

If $|a|<1$ then $|ae^{-jw}|<1$ and we can expand the log as a power series using

$$\log(1-d) = -\left(d + \frac{d^2}{2} + \frac{d^3}{3} + \ldots\right) \quad \text{for} \quad |d| < 1$$

Hence $$\log\left(H\left(e^{j\omega}\right)\right) = -\sum_{n=1}^{\infty} \frac{a^n}{n} e^{-jn\omega}$$

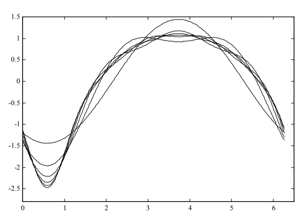
Note that $\log\left(re^{j\phi}\right) = \log(r) + j\phi \;\Rightarrow\; \log(|x|) = \Re(\log(x))$

Hence

$$\log\left(\left|H\left(e^{j\omega}\right)\right|^2\right) = -2\sum_{n=1}^{\infty} \frac{r^n}{n} \cos(n(\omega-\theta)) \quad \text{where} \quad a = re^{j\theta}$$

First six terms in the summation for:

$a = 0.6 + 0.4j$




The average of $\log|H|^2$ is always zero if $|a|<1$



## Log Frequency Response for |a|>1

If $|a|>1$, we can rearrange the formula in terms of $a^{-1}$:

$$\log\left(H(e^{j\omega})\right) = \log\left(-ae^{-j\omega}\left(1 - a^{-1}e^{j\omega}\right)\right)$$
$$= \log\left(-ae^{-j\omega}\right) + \log\left(1 - a^{-1}e^{j\omega}\right)$$

Since $|a^{-1}|<1$, we can expand the log as before to obtain

$$\log\left(\left|H\left(e^{j\omega}\right)\right|^2\right) = 2\log|a| - 2\sum_{n=1}^{\infty} \frac{r^{-n}}{n} \cos(n(\omega-\theta))$$
$$\text{where} \quad a = re^{j\theta}$$

The average of $\log|H|^2$ is $2\log|a|$ if $|a|>1$

The log response of an arbitrary filter is just the sum of the log responses of each pole or zero. For a stable filter, all the poles must be within the unit circle. Hence ...

Given a stable filter: $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots}$

then the average value of $\log|H|^2$ is given by

$$2\log\left(\left|\frac{b_0}{a_0}\right|\right) \quad + 2 \sum_{\text{zeros with } |z|>1} \log(|z_i|)$$

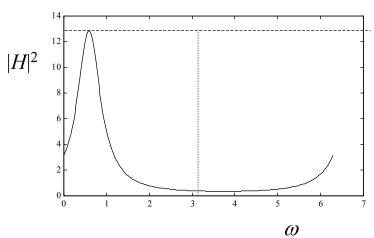
## Single pole filter

$$H(z)=\frac{1}{1-az^{-1}} \quad \Leftrightarrow \quad y(n)=x(n)+ay(n-1)$$

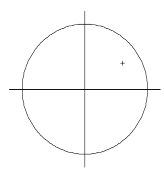
Filter has a single pole at $z = a = re^{j\theta}$.

Power response of filter is given by:

$$\left|H(e^{j\omega})\right|^2=\frac{1}{1+r^2-2r\cos(\omega-\theta)}$$




Peak = $(1-r)^{-2}$



Note: response is no longer a simple cosine wave



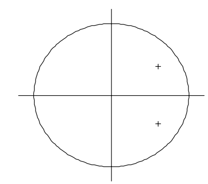
## Pole Pairs

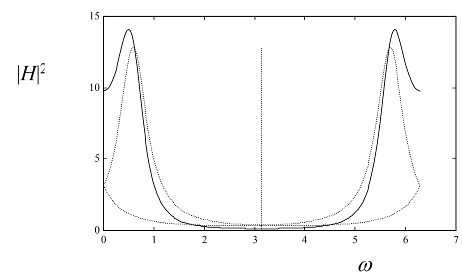
If the filter coefficients are real, any complex zeros or poles will always occur in conjugate pairs.

The response of the filter is the product of the responses of the individual poles. Conjugate pole/zero pairs ensure a symmetric response.

Example: Poles at $0.6 \pm 0.4j = 0.72e^{\pm 0.59j} = re^{\pm j\theta}$

$$H(z)=\frac{1}{1-2r\cos\theta z^{-1}+r^2z^{-2}} = \frac{1}{1-1.2z^{-1}+0.52z^{-2}}$$

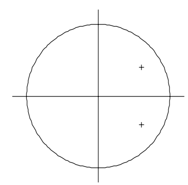




## Geometrical Interpretation

$$H(z)=\frac{1}{(1-az^{-1})(1-a^{*}z^{-1})}$$

$$|H(z)|=\frac{1}{|1-az^{-1}||1-a^{*}z^{-1}|}$$



But since |z|=1, we have $|1-az^{-1}|=|z^{-1}||z-a|=|z-a|$

This is just the distance between z and a.

The magnitude response of the filter

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\ldots}{a_0+a_1z^{-1}+a_2z^{-2}+\ldots}=\frac{b_0}{a_0}\times\frac{\prod(1-x_iz^{-1})}{\prod(1-y_iz^{-1})}$$

at a frequency ω is proportional to the product of the distance from the point $e^{j\omega}$ to all the zeros divided by the product of the distance to all the poles .

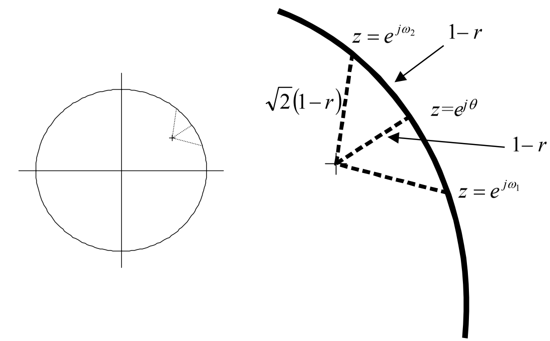
The constant of proportionality is $b_0/a_0$.

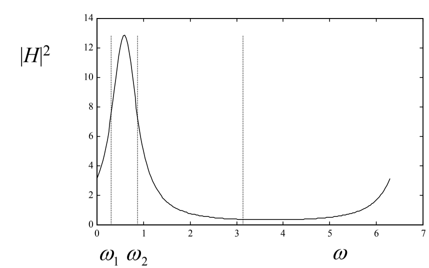


## Bandwidth of a Resonance Peak

The bandwidth of a resonance peak is the frequency range at which the magnitude response has decreased by √2.

For poles near the unit circle this is approximately $2(1-r)$ rad/s = $(1-r)/\pi$ Hz (normalised).




$$\omega_2-\omega_1\approx 2(1-r)$$

## Bandwidth expansion

If we have a filter

$$H(z)=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\dots}{a_0+a_1z^{-1}+a_2z^{-2}+\dots}$$

We can form a new filter by multiplying coefficients $a_i$ and $b_i$ by $k^i$ for some $k < 1$.

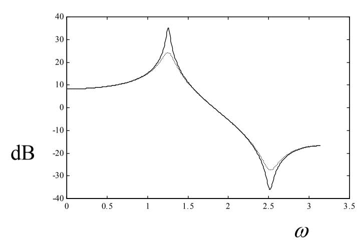
$$G(z)=H(z/k)=\frac{b_0+b_1kz^{-1}+b_2k^2z^{-2}+\dots}{a_0+a_1kz^{-1}+a_2k^2z^{-2}+\dots}$$

If $H(z)$ has a pole/zero at $z_0$, then $G(z)$ will have one at $kz_0$. All poles and zero will be moved inwards by a factor $k$.

If the bandwidth of a pole of $H(z)$ is $b$=2(1–$r$), then the bandwidth of the corresponding pole in $G(z)$ will be expanded to: $$2(1-kr)=b+2r(1-k)$$

$$\frac{1+1.57z^{-1}+0.94z^{-2}}{1-0.67z^{-1}+0.96z^{-2}}$$

$$k=0.95$$






## Coefficient Reversal

If we have a filter

$$H(z)=b_0+b_1z^{-1}+b_2z^{-2}+\dots+b_pz^{-p}$$

We can form a new filter by conjugating the coefficients and putting them in reverse order:

$$G(z)=b_p^*+b_{p-1}^*z^{-1}+b_{p-2}^*z^{-2}+\dots+b_0^*z^{-p}=z^{-p}H^*\left(z^{*-1}\right)$$

If $z_0$ is a zero of $H(z)$ then $z_0{}^{*-1}$ is a zero of $G(z)$. This is called a *reflection* in the unit circle.

The frequency response of G(z) is given by:

$$G(e^{j\omega})=e^{-jp\omega}H^*(e^{j\omega})$$

Hence G(z) has the same magnitude response as H(z) but a different phase response:

$$\left|G(e^{j\omega})\right|=\left|H(e^{j\omega})\right| \qquad \mathrm{Arg}\left(G(e^{j\omega})\right)=-\mathrm{Arg}\left(H(e^{j\omega})\right)-p\omega$$