RESOLVING NEAR-CARRIER SPECTRAL INFINITIES DUE TO $1/F$ PHASE NOISE IN OSCILLATORS

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ABSTRACT

In this paper, we derive an expression for the near-carrier power spectral density of an oscillator having $1/f$ phase noise. Motivated by empirical metrics such as the Allan variance, we develop a rigorous mathematical analysis and derive a closed-form expression for the oscillator autocorrelation function in the case of exactly $1/f$ phase noise that is smoothed using a rectangular time window. We show that this smoothed $1/f$ phase noise results in a finite variance noise process and preserves oscillator stationarity. Furthermore, in agreement with experimental data, we explain how a quadratic and a logarithmic term appear in the autocorrelation function and establish the relationship between the logarithmic term and the $1/f$ characteristics of the oscillator random process.

***Index Terms*—** flicker noise, phase noise, power spectral density, oscillators, stationarity

1. INTRODUCTION

It is known that numerous physical fluctuations have empirical (measured) spectral densities proportional to $1/f^{\alpha}$ with α in the vicinity of 1 ($\alpha = 1 \pm \nu, \nu \ll 1$). This spectral behavior results in two serious difficulties with finite power systems analysis. Infinite power is implied due to both low and high frequencies tendency to approach dc[1] and ∞ at a $1/f$ rate.

In terms of the high frequency infinity, a widely accepted assumption is that there has to exist a high frequency f_h, at which the PSD slope becomes steeper [7]. It has never been possible to measure such a high corner frequency, because $1/f$ noise disappears in the white thermal noise that is omnipresent in all electronic systems. However, as Plank's analysis for thermal noise revealed a higher cut-off frequency in the case of white noise sources, the above assumption appears plausible.

As a result of the above, research interest has mainly focused in characterizing the nature of $1/f$ perturbations near dc. As with the high frequencies case, the concept of a lower cut-off frequency has been introduced, below which the PSD is expected to flatten off. This approach was adopted in [5] for the analysis of $1/f$ noise sources in the phase of an oscillator. The resulting expression for the oscillator autocorrelation function was used to approximate the $1/f$ region as a Lorentzian power spectral density (PSD).

Generally, in the past, theoretical models for $1/f$ processes either proposed its representation as a non-stationary random process (RP) that exhibits infinite memory [8], or concluded that the so-called "infrared catastrophe" cannot be tackled with traditional statistical tools [11]. Work by Handel in [6] examines the origins of $1/f$ noise in electronic devices in the general context of quantum $1/f$ processes. In this work he argues that the PSD of $1/f$ processes diverges to a characteristic with a slope smaller that unity near dc, so that the "infrared catastrophe" described in [11] is overcome.

In the present work we avoid making the assumption of a lower cut-off frequency for two reasons: i) such a lower cut-off frequency has never been observed in experimental data and ii) there does not exist a theoretical value for the determination of such a lower cut-off frequency. However, in order to avoid infinities in the phase noise variance, we will assume a "smoothed" version of $1/f$ phase noise, motivated by widely used practical metrics such as the Allan variance. Following this approach, we derive closed form expressions for the oscillator autocorrelation function and for the near-carrier PSD. We show that smoothed $1/f$ noise does not perturb the oscillator stationarity.

2. PHASE NOISE AND FREQUENCY FLUCTUATIONS IN OSCILLATORS

A rich nomenclature exists concerning noisy oscillators, reflecting the efforts to combine the conclusions of empirical analyses on one hand with abstract models and theory on the other. In the present study, we only need to define a noisy oscillator at a frequency ω_{osc} as a system whose time domain output $\zeta(t)$ can be described as:

$$\zeta(t) = (1 + \varepsilon(t)) \cos(\omega_{osc} t + \phi(t)) \quad (1)$$

where $\varepsilon(t)$ and $\phi(t)$ are real RPs[2]. In published research it has been shown that amplitude fluctuations expressed through $\varepsilon(t)$ have a negligible effect in the near-carrier PSD that interests us and can therefore be neglected [13], [9]. Furthermore, it is shown in [2] that

[1] For convenience we use dc in this paper to denote the nominal or average oscillation frequency.

[2] The use of a sinusoidal oscillator does not restrict the generality of our approach as any periodic function can be analyzed in a cosine-based Fourier series.

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the PSD of the real valued oscillator $\zeta(t)$ can be reconstructed by halving the PSD of its complex valued analytic[3] version:

$$\psi(t) = e^{j(\omega_{osc}t+\phi(t))} \tag{2}$$

assuming $\varepsilon(t) = 0$.

Using the frequency translation property of the Fourier Transform $\mathcal{F}(\cdot)$ we can proceed in our analysis by considering solely the complex valued RP

$$w(t) = e^{j\phi(t)}. \tag{3}$$

We would like to calculate the autocorrelation function of the noise RP $w(t)$ in order to determine the expression for the oscillator noise spectrum using the Wiener-Khinchin theorem, given that the oscillator can be modeled as a wide sense stationary (WSS) process. Therefore, we are interested in evaluating the expression

$$\begin{aligned} R_{ww}(\tau) &= \mathbf{E}[w(t)w^*(t-\tau)] \\ &= \mathbf{E}[e^{j(\phi(t)-\phi(t-\tau))}] \\ &= \mathbf{E}[e^{j\xi(t,\tau)}] \end{aligned} \tag{4}$$

where $\mathbf{E}[\cdot]$ denotes statistical expectation. It is important to note that the phase noise variation RP $\xi(t,\tau)$

$$\xi(t,\tau) = \phi(t) - \phi(t-\tau) \tag{5}$$

that represents 1st order phase noise differences emerges naturally in the expression of the oscillator autocorrelation function without imposing any assumptions. In practice, a large number of important empirical metrics [1] make indirect use of the phase noise variation RP $\xi(t,\tau)$ by utilizing it in empirical estimates of the phase noise variance of finite length measurement sets.

In order to evaluate $R_{ww}(\tau)$ we have to make some assumption about the phase noise process statistics. Assuming that the RP $\phi(t)$ is zero-mean-Gaussian (ZMG) and that the RPs $\phi(t)$ and $\phi(t-\tau)$ are jointly Gaussian, it follows that $\xi(t,\tau)$ is ZMG. In that case, the autocorrelation of the oscillator noise is simply given as

$$R_{ww}(\tau) = e^{-\frac{\sigma_\xi^2(\tau)}{2}} \tag{6}$$

where $\sigma_\xi^2(\tau)$ denotes the variance of the RP $\xi(t,\tau)$. Modeling $\phi(t)$ as the integral of frequency fluctuations $\Omega(t)$ whose PSD is denoted $S_\Omega(\omega)$, we have that $S_\Omega(\omega) = \omega^2 S_\phi(\omega)$, with $S_\phi(\omega)$ denoting the PSD of $\phi(t)$. We can express $\xi(t,\tau)$ as a function of frequency fluctuations as

$$\xi(t,\tau) = \int_t^{t+\tau} \Omega(t')dt', \tag{7}$$

so that

$$\begin{aligned} \sigma_\xi^2(\tau) &= \mathbf{E}[\xi(t,\tau)^2] \\ &= \frac{2}{\pi}\int_0^\infty S_\Omega(\omega)\frac{1-\cos(\omega\tau)}{\omega^2}\mathrm{d}\omega \\ &= \int_0^\infty S_\Omega(2\pi f)\frac{\sin^2(\pi f\tau)}{\pi^2 f^2}\mathrm{d}f \end{aligned} \tag{8}$$

A detailed derivation of (8) can be found in [2]. A key point resulting from the above is that 1st order differences in the phase noise process $\phi(t)$ are equivalent to multiplication of the frequency noise PSD $S_\Omega(\omega)$ with a sinc^2 low pass filter (LPF) as far as the evaluation of $\sigma_\xi^2(\tau)$ is concerned.

[3]Related through the Hilbert Transform.

Based on this result, Chorti and Brookes were able to derive closed-form expressions for the noise RP $w(t)$ autocorrelation function $R_{ww}(\tau)$ and its PSD[4] $S_w(\omega)$ [2] in the following important cases:

1. Phase modulated white phase noise with PSD k_0. The noise RP PSD $S_w(\omega)$ is a Dirac Delta weighted by $2\pi e^{-k_0\omega_B/2}$, where ω_B is the bandwidth of the bandlimited white-like phase noise, combined with a flat region with PSD k_0 up to ω_B.

2. Phase modulated approximately $1/f$ phase noise with PSD $k_1/|f|^{1+\nu}$, $0 < \nu \ll 1$. The noise RP PSD can be expressed as the infinite sum of sub-spectra, in the form

$$S_w(\omega) = 2\pi\delta(\omega) + \sum_{n=1}^{\infty} C_n(\nu,k_1)|\omega^{-1-n\nu}| \tag{9}$$

with

$$C_n(\nu,k_1) = (-1)^{n+1}\frac{2\sin(\frac{\pi n\nu}{2})\nu(\frac{2\gamma_1 k1}{\nu})^n\Gamma(n\nu)}{\Gamma(n)} \tag{10}$$

where $\Gamma(\cdot)$ denotes the Gamma function. For sufficiently small ν, the dominant sub-spectral term is for $n = 1$:

$$\widetilde{S}_w(\omega) = \frac{4\sin(\frac{\pi\nu}{2})\gamma_1 k_1\Gamma(\nu)}{|\omega^{1+\nu}|}. \tag{11}$$

More importantly, the value of the PSD on the carrier is finite and equals:

$$S_w(0) = 8\pi\frac{\Gamma(\frac{1}{\nu})}{\nu}\sqrt[\nu]{-\frac{\Gamma(\frac{1+\nu}{2})}{\sqrt{\pi}k_1\Gamma(-\frac{\nu}{2})}}. \tag{12}$$

3. Frequency modulated white phase noise with PSD k_2/f^2. The oscillator is a Brownian motion process [12] with a Lorentzian PSD.

4. Frequency modulated approximately $1/f$ phase noise with PSD $k_3/|f|^{3-\nu}$, $0 < \nu \ll 1$. The analysis of Klimovitch [10] has shown that the near-carrier PSD can be approximated by a Gaussian region followed by a $1/|f|^3$ region,

$$S_w(\omega) = \frac{\sqrt{2\pi}}{\sqrt{K}}e^{-\frac{\omega^2}{2K}} + \frac{8\pi^3 k_3}{|\omega|^3}u(|\omega|-\omega_3) \tag{13}$$

with

$$K = \frac{16\pi^2\sqrt{\pi}(2\pi)^{-\nu}\Gamma(\frac{\nu}{2})k_3}{(2-\nu)\Gamma(\frac{3}{2}-\frac{\nu}{2})}. \tag{14}$$

f_3 is the frequency of transition from the Gaussian to the power-law region and $u(\cdot)$ denotes the Heaviside function.

5. Frequency modulated random walk phase noise of PSD k_4/f^4. The oscillator PSD is a Gaussian in the near carrier frequencies followed by a power-law region at the frequency of transition ω_4:

$$S_w(\omega) = \frac{\sqrt{\rho}}{2\pi\sqrt{\pi k_4}}e^{-\frac{\rho\omega^2}{16\pi^4 k_4}} + \frac{16\pi^4 k_4}{\omega^4}u(|\omega|-\omega_4) \tag{15}$$

[4]The oscillator PSD $S_\zeta(\omega)$ can be obtained by convolution of $2\pi\delta(\omega-\omega_{osc})$ with the PSD of the RP $w(t)$.

As noted above, time averages of 1st order phase noise differences appear naturally in the oscillator autocorrelation function, imposing a smoothing of the phase noise high frequency components. Equivalently, the oscillator loop acts as a sinc^2 low-pass filter (LPF) for frequency fluctuation samples. As a result, under a joint Gaussianity assumption for the phase noise samples, we are able to derive closed-form expressions for the oscillator autocorrelation function and PSD accounting for the main even order power-law phase noise processes, k_0, k_2/f^2, k_4/f^4.

However, in the case of $1/f$ phase noise (and the related $1/f^3$), the 1st order phase noise sample differences do not offer sufficient smoothing for controlling the induced infinities and the related variance integrals explode to infinity. However, it was readily shown in [2] and [10] that approximately $1/f$ and $1/f^3$ phase noise is not incompatible with oscillator stationarity and that we can obtain closed-form expressions for the relevant oscillator autocorrelation functions and PSDs. Although such processes have infinite variance, the oscillator behavior as a limit cycle in the state space [4] can "absorb" infinities of a rate close but not equal to $1/f$. An interesting question in that sense would be to identify the necessary degree of smoothing of the $1/f$ phase noise in order to avoid the aforementioned infinities; what kind of $1/f$ noise can be "absorbed" by the oscillator loop before the latter becomes nonstationary?

3. RESOLVING INFINITIES IN THE CASE OF EXACTLY $1/F$ NOISE IN OSCILLATORS

In the previous section we have discussed how 1st order phase noise sample differences or equivalently a sinc^2 LPF of the frequency fluctuations RP is not sufficient to avoid infinities in the case of exactly $1/f$ noise. As a result, 2nd order phase noise differences were used in practise to produce a set of useful metrics such as the Allan variance.

In the following, we propose an equivalent approach by performing a smoothing of the RP $\Omega(t)$ of frequency fluctuations. As a result, a specific type of $1/f$ noise in oscillators will be examined. We consider a phase noise process $\psi(t)$ that is generated by frequency fluctuations $\Psi(t)$ by convolving $\Omega(t)$ with a rectangular time window of length $2T$. This will result in a further sinc^2 filtering of the frequency fluctuations, that is analogous to 2nd order phase noise sample differences, so that:

$$S_\Psi(f) = k_1|f|\frac{\sin^2(fT)}{f^2} \tag{16}$$

which corresponds to

$$S_\Psi(\omega) = 4\pi^3 k_1 \frac{\sin^2(\omega T)}{|\omega|} \tag{17}$$

for energy conservation while $S_\Psi(\omega)$ corresponds to the PSD of $\Psi(t)$. The process $\xi(t,\tau,T)$ now depends on the length T of the time window. Following a similar analysis to that included in section II we obtain:

$$\begin{aligned}
\sigma_\xi^2(\tau,T) &= \mathbf{E}[\xi(t,\tau,T)^2] \\
&= \frac{2}{\pi}\int_0^\infty S_\Psi(\omega)\frac{1-\cos(\omega\tau)}{\omega^2}\mathrm{d}\omega \\
&= \frac{2}{\pi}\int_0^\infty 4\pi^3 k_1 \frac{\sin^2(\pi f\tau)}{2\pi|f|}\frac{\sin^2(\pi f T)}{4\pi^2 f^2}\mathrm{d}2\pi f \\
&= 2k_1\Big[(\tau+T)^2\ln((\tau+T)^2) \\
&+ (\tau-T)^2\ln((\tau-T)^2) \\
&- 2\tau^2\ln(\tau^2) - 2T^2\ln(T^2)\Big]
\end{aligned} \tag{18}$$

The above expression for the variance of the phase noise variation RP is in agreement with experimental data that find an approximately quadratic dependence on τ [1]. Demir in [3] provides a qualitative explanation of this quadratic dependence. However, it is worth noting that the multiplication of the quadratic terms with a logarithmic term weakens the possibility of obtaining a Lorentzian PSD close to the carrier as proposed by Demir in [5]. In fact, in the following we will see that it is the dependence of $\sigma_\xi^2(t,T)$ on $\ln(\tau^2)$ that provokes the $1/f$ behavior of the oscillator output at measurable offset frequencies.

As depicted in (18) the variance of the process $\xi(t,\tau,T)$ is a function of τ, T. Without loss of generality we can obtain a simplified expression if we set $T = 1$, that is if we scale "time" to the length of the frequency fluctuations observation time window. The expression for the oscillator phase noise autocorrelation function then becomes

$$\begin{aligned}
R_{ww|T=1}(\tau) &= \exp(-2k_1((\tau+1)^2\ln((\tau+1)^2 \\
&+ (\tau-1)^2\ln((\tau-1)^2)) - 2\tau^2\ln(\tau^2)))
\end{aligned} \tag{19}$$

It is not possible to find the Fourier transform of the above expression analytically. An additional problem with the numerical calculation of the PSD is that as stands above, $R_{ww|T=1}$ is not a rapidly decreasing function of time and the integration interval has to be very large resulting into unreasonable computation times. However, we can simplify things if we use the following reasoning; the near-carrier PSD corresponds to the dominant components of $R_{ww|T=1}$ as $\tau\to\infty$. Simply taking the limit of (19) as $\tau\to\infty$ is not useful as the latter is null, as expected for any real autocorrelation function. Nevertheless, we can isolate the terms including the effect of $\ln(\tau^2)$ if rewrite the variance of the RP $\xi(t,\tau)$ as

$$\begin{aligned}
\sigma_{\xi|T=1}^2(\tau) &= 2k_1\Big[(\tau+1)^2\ln(\tau^2(1+\tau^{-1})^2) \\
&+ (\tau-1)^2\ln(\tau^2(\tau-\tau^{-1})^2) - 2\tau^2\ln(\tau^2)\Big] \\
&= 2k_1\Big[(\tau+1)^2\ln(\tau^2) + (\tau-1)^2\ln(\tau^2) \\
&- 2\tau^2\ln(\tau^2) + (\tau+1)^2\ln((1+\tau^{-1})^2) \\
&+ (\tau-1)^2\ln((1-\tau^{-1})^2)\Big].
\end{aligned} \tag{20}$$

By approximating $\ln(\cdot)$ with a 3-term Taylor series we find that

$$\lim_{\tau\to\infty}(\tau+1)^2\ln((1+\tau^{-1})^2) + (\tau-1)^2\ln((1-\tau^{-1})^2) = 6 \tag{21}$$

We can therefore approximate (20) with

$$\sigma_{\xi|T=1}^2(\tau) = 2k_1(2\ln(\tau^2)+6) = 2k_1\big(\ln(\tau^4)+6\big) \tag{22}$$

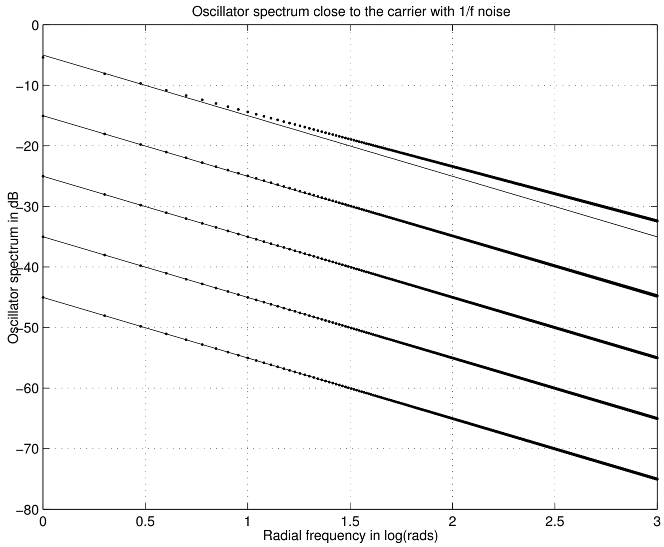

Fig. 1. Estimated (points) $k_1/f^{1-\gamma}$ and theoretical (solid lines) k_1/f PSD of an oscillator with $1/f$ phase noise. From top to bottom $[k_1 = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}]$ Watt/rad.

and we obtain the following approximation for the autocorrelation function for large τ

$$\widetilde{R}_{ww}(\tau) = e^{-k_1\left(\ln(\tau^4)+6\right)} \tag{23}$$

The spectrum of (23) can be analytically found to be

$$\widetilde{S_w}(\omega) = \frac{e^{-6\gamma}2^{-\gamma}\sqrt{\pi}\Gamma(\frac{1-\gamma}{2})}{\Gamma(\frac{\gamma}{2})|\omega|^{1-\gamma}} \tag{24}$$

where

$$\gamma = 4k_1 \tag{25}$$

As $k_1 << 1$, we can approximate

$$e^{-6\gamma}2^{-\gamma} \simeq 1 \tag{26}$$

$$\Gamma\left(\frac{1-\gamma}{2}\right) \simeq \sqrt{\pi} \tag{27}$$

$$\Gamma\left(\frac{\gamma}{2}\right) \simeq \frac{2}{\gamma}. \tag{28}$$

Therefore (24) can be approximated as

$$\widetilde{S_w}(\omega) \simeq \frac{2\pi k_1}{|\omega|^{1-\gamma}}. \tag{29}$$

Therefore, the oscillator is predicted to have an approximately $k_1/|f|$ spectrum, as shown in Fig. 1.

The PSD is infinite on the carrier frequency, however, the oscillator power around the carrier over a bandwidth $2\omega_B$ is finite as the integral of (29) converges:

$$\frac{1}{2\pi}\int_{-\omega_B}^{\omega_B} \widetilde{S_w}(\omega)\mathrm{d}\omega = \frac{1}{2}\omega_B^{\gamma} \tag{30}$$

As a result, 2nd order differences of $1/f$ phase noise samples provide the means of resolving infinities in flicker phase noise analysis.

4. CONCLUSIONS

In this paper we have studied the behavior of an oscillator with "smoothed" $1/f$ phase noise. We have generated $1/f$ phase noise from the integration of the convolution of frequency fluctuations with a rectangular time window to generate the equivalent of 2nd order differences in phase noise samples. An important conclusion of our analysis is that when non-stationary flicker noise is observed over a finite time window in the phase of a real oscillator, the resulting RP possesses finite power and is WSS. The exact spectrum can be found as the sum of an approximately $1/f$ spectrum as expressed in (24) and the Fourier transform of the deviation of (23) from (19). It still remains a question how best to estimate this second term in the spectrum, given that numerical calculations have to be extended to very large time intervals.

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