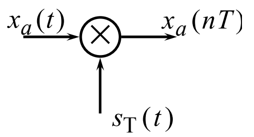
Module 1

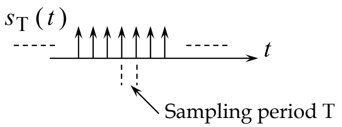
Sampling and Aliasing, System Functions
and z-transforms

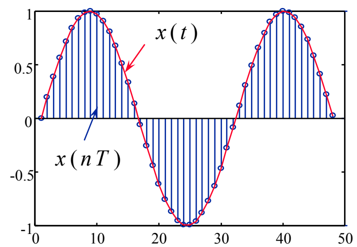
Contents

- Sampling and Aliasing
  - Concept of sampling a continuous-time signal
  - Periodic nature of discrete-time signals in the frequency domain, the effect of aliasing and Nyquist sampling criterion
  - Reconstruction of continuous-time signals from their samples
- System Functions
  - Transfer function
  - Frequency response
  - Example
- z-transforms
  - Definition and properties
  - Inverse z-transform
  - Causal and anticausal systems
  - Stability

Sampling







- Analog signal $x_a(t)$
- Sampling function $s_T(t)$ is a sequence of impulses

$$s_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

Symbols

- continuous time signal $x_a(t)$
- samples of continuous time signal $x_a(nT)$
- discrete-time signal $x(n)$
- frequency Ω
- digital frequency ω
- *FT* of $x_a(t)$ $X_a(\Omega)$
- *DFT* of $x(n)$ $X(e^{j\omega})$

Analysis of Sampled Signal

- Spectrum of discrete-time signal = spectrum of continuous-time signal + images at multiples of 2π
  - From *IFT*:

$$x(n) = x_a(nT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega nT}d\Omega$$

  - From *IDFT*:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

  - Combining these:

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right)$$

- Proof
  - write expression for $x_a(nT)$ as sum of integrals over intervals of length $2\pi/T$

$$x(n) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{(2r-1)\pi/T}^{(2r+1)\pi/T} X_a(j\Omega)e^{j\Omega nT}d\Omega$$

  - change of variable: replace Ω with $\Omega' + 2\pi r/T$

$$x(n) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty}\int_{-\pi/T}^{\pi/T} X_a\left(j\left(\Omega'+\frac{2\pi r}{T}\right)\right)e^{j\Omega' nT}e^{2\pi nr}d\Omega'$$

  - use $e^{j2\pi rn} = 1 \quad \forall(r,n)$ integer, reverse order of sum and integration and use $\Omega' = \omega/T$

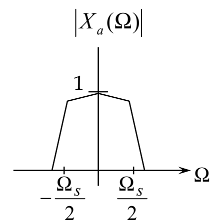
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right)\right]e^{j\omega n}d\omega$$

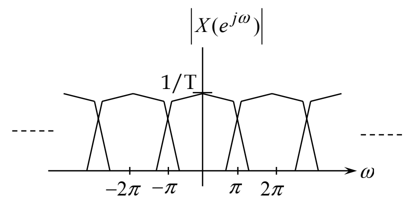
$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right)\right]e^{j\omega n}d\omega$$

  - Note that this is in the form of an IDFT since
  - Hence

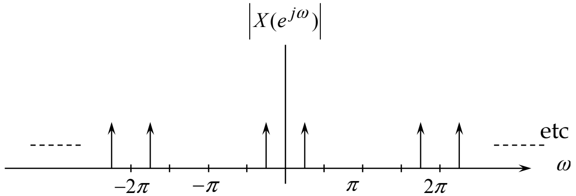
$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_a\left(j\left(\frac{\omega}{T}+\frac{2\pi r}{T}\right)\right) \qquad [1]$$

- Points to note:
  - spectrum of *x(n)* is periodic in ω with period 2π
  - if $X_a(\Omega)$ is not bandlimited to $\Omega_s/2$ then information in the signal is lost when sampled due to overlapping spectral images - this effect is called <u>aliasing</u>
  - if $X_a(j\Omega)$ is bandlimited to $\Omega_s/2$ then the original continuous-time signal can be perfectly reconstructed from its discrete-time samples
    - this is known as the Nyquist Sampling Criterion
  - Ω is the analog frequency, $\Omega = 2\pi f \quad 0 < \Omega < \infty$
  - ω is the digital frequency $\omega = \Omega T = 2\pi f T = \frac{2\pi f}{f_s}$

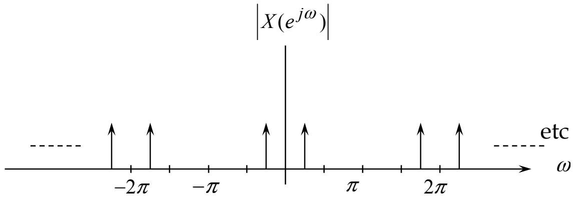




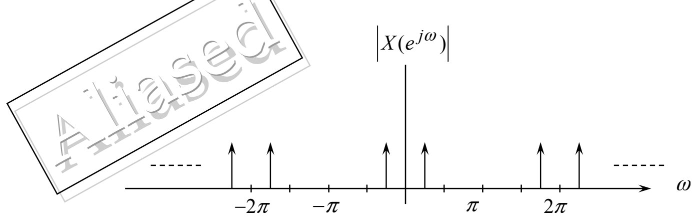
- Examples of signal spectra after sampling
  - 1) sinusoidal signal at 1 kHz, sampling frequency = 8 kHz



  - 2) sinusoidal signal at 5.5125 kHz, sampling frequency = 44.1 kHz



  - 3) sinusoidal signal at 1 kHz, sampling frequency = 1.1429 kHz



- Sampling in time domain => periodicity in frequency

- Sampling in frequency domain => periodicity in time

Signal Reconstruction

- A continuous-time signal $x_a(t)$ can be reconstructed from its samples $\{x_a(nT)\}$ as

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) g(t - nT)$$

  - where $g(t) = \dfrac{\sin(\pi t/T)}{\pi t/T}$

    and corresponds to a lowpass filter with cut-off at the Nyquist frequency.

- Proof
  - write the *IFT* expression for $x_a(t)$ for the range $-\pi/T \le \Omega \le \pi/T$ or equivalently $-\pi \le \omega \le \pi$

$$x_a(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X_a(\Omega) e^{j\Omega t} d\Omega$$

  - from [1] we know that, in the range $-\pi \le \omega \le \pi$

$$X(e^{j\omega}) = X(e^{j\Omega T}) = \frac{1}{T} X_a(\Omega)$$

  - giving

$$x_a(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} T X(e^{j\Omega T}) e^{j\Omega t} d\Omega$$

- Proof (continued)
  - using the DTFT relation (described later)

$$X(e^{j\Omega T}) = \sum_{n=-\infty}^{\infty} x_a(nT) e^{-j\Omega nT}$$

  - write

$$x_a(t) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{n=-\infty}^{\infty} x_a(nT) e^{-j\Omega nT} \right] e^{j\Omega t} d\Omega$$

  - change order of summation and integration

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT) \left[\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} e^{j\Omega(t-nT)} d\Omega \right]$$

$$= \sum_{n=-\infty}^{\infty} x_a(nT) \frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}$$

- Proof (continued)
  - This operation can be recognized as the convolution of $x_a(nT)$ with the sinc function

$$\frac{\sin(\pi t/T)}{\pi t/T}$$

  - This convolution represents filtering with an "ideal" lowpass filter with a cut-off frequency of $\omega = \pi$
    - the Nyquist frequency

Reading: Proakis: Chapter 1, especially 1.4.1 to 1.4.7

System Functions

- Transfer function
  - For a continuous-time system H(s) with input X(s) and output Y(s),

    its transfer function is defined as $H(s) = \dfrac{Y(s)}{X(s)}$

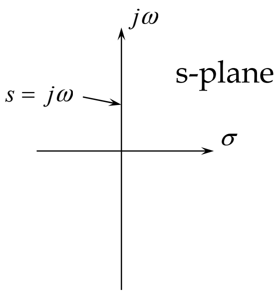
  - For a discrete-time system H(z) with input X(z) and output Y(z),

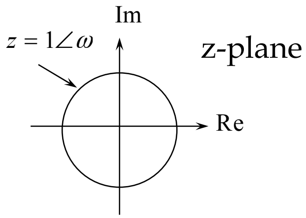
    its transfer function is defined as $H(z) = \dfrac{Y(z)}{X(z)}$

    - *H(.), Y(.)* and *X(.)* are polynomials in (.)

Frequency response

- For continuous-time systems, use $s = \sigma + j\omega$ and investigate the function *H(s)* as a function of frequency ω only, i.e. write $s = j\omega$
- For discrete-time systems, use $z = e^{sT}$, $s = \alpha + j\omega$ and investigate the function *H(z)* as a function of frequency ω only, i.e. write $z = e^{j\omega}$

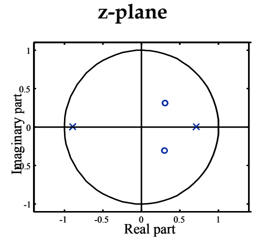




Example:

- Transfer Function

$$H(z) = \frac{z^2 - 0.6z + 0.18}{z^2 + 0.2z - 0.63}$$

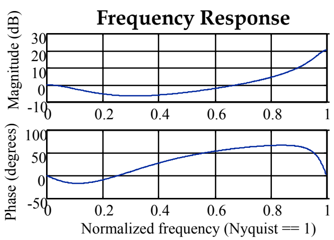


- zeros at $z = 0.3+j0.3$ and $z = 0.3-j0.3$
- poles at $z = -0.9$ and $z = 0.7$

- Can be written in terms of z^{-1} as $H(z) = \dfrac{1 - 0.6z^{-1} + 0.18z^{-2}}{1 + 0.2z^{-1} - 0.63z^{-2}}$
- Difference Equation:

$$y(n) = x(n) - 0.6x(n-1) + 0.18x(n-2) - 0.2y(n-1) + 0.63y(n-2)$$

- Frequency Response
  - set $z = e^{j\omega}$
  - plot magnitude and phase



  - normally plot for $0 < \omega < \pi$ normalized such that $\pi = 1$

Reading: Proakis, Chapter 2 especially 2.4.and 2.5

z-transform

Definition

- The z-transform of the sequence *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The *z* domain for discrete-time signals is analogous to the *s* domain for continuous-time signals
- the *z* domain allows a signal (or system) to be compactly described

Notation

- Z-transform denoted by

$$X(z) \equiv Z\{x(n)\}$$

- relationship indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

Region of Convergence

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- z-transform is an infinite power series
  - only exists for particular values of z for which the series converges
  - these are the values of z for which $X(z)$ has a finite value
- need to specify Region Of Convergence (ROC) when referring to z-transform

Examples

- Finite Duration Sequences

$x(n) = \{1, 2, 3, 5, 8\}$ — $X(z) = 1 + 2z^{-1} + 3z^{-2} + 5z^{-3} + 8z^{-4}$

ROC: $z \neq 0$

$x(n) = \{1, 2, \underset{\uparrow}{3}, 5, 8\}$ — $X(z) = z^{2} + 2z^{1} + 3 + 5z^{-1} + 8z^{-2}$

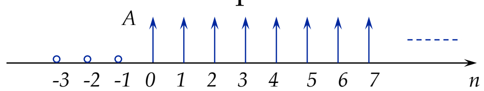
ROC: $z \neq 0$ and $z \neq \infty$

$x(n) = \delta(n)$ — $X(z) = 1$

ROC: $\forall z$

- Infinite Duration Sequences

Step Function



Recall that an infinite geometric series

$$1 + A + A^2 + \ldots = \frac{1}{1-A}$$

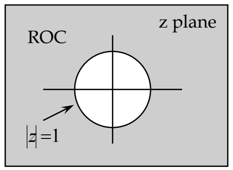
if $|A| < 1$

$$x(n) = Au(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} Au(n) z^{-n}$$

$$= A\left(1 + z^{-1} + z^{-2} + z^{-3} + \cdots\right)$$

$$= \frac{A}{1 - z^{-1}} \quad \text{ROC: } |z^{-1}| < 1$$



- Sequences like $Au(n)$ are called "right-sided" sequences

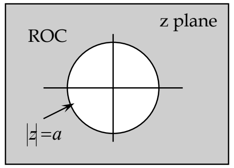
Another Example

$$x(n) = Au(n)a^n e^{jn\omega_1}$$

$$X(z) = \sum_{n=-\infty}^{\infty} Au(n)a^n e^{jn\omega_1} z^{-n}$$

$$= A\sum_{n=0}^{\infty}\left(ae^{j\omega_1}z^{-1}\right)^n$$

$$= \frac{A}{1-ae^{j\omega_1}z^{-1}}$$



- The region of convergence is

$$\left|ae^{j\omega}z^{-1}\right| < 1$$

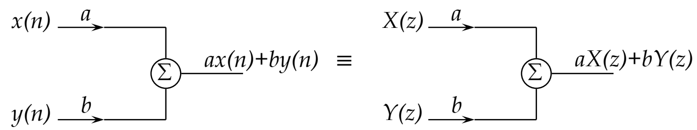
or $|z| > |a|$

since $\left|e^{j\omega}\right| = 1$

Properties of the z-transform

- Linearity
  - for signals $x(n)$ and $y(n)$ with z-transforms $X(z)$ and $Y(z)$

$$\sum_{k=1}^{p} C_k f_k(n) \leftrightarrow \sum_{k=1}^{p} C_k F_k(z)$$



- Shift

$$Z\{f(n-m)\} = z^{-m}F(z)$$

- Multiplication by n

$$Z\{nf(n)\} = -z\frac{dF(z)}{dz}$$

  - Proof $\quad F(z) = \sum_{n=-\infty}^{\infty} f(n)z^{-n}$
    - differentiating both sides gives

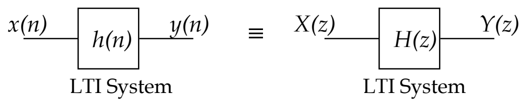
$$\frac{dF(z)}{dz} = \sum_{n=-\infty}^{\infty} -nf(n)z^{-(n+1)}$$

    - multiplying both sides by -z gives

$$-z\frac{dF(z)}{dz} = \sum_{n=-\infty}^{\infty} nf(n)z^{-n}$$

    - where the RHS can be seen to be the z-transform of *nf(n)*

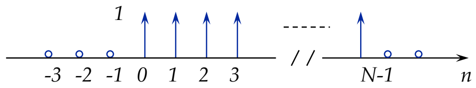
- Convolution



$$y(n) = \sum_{m=-\infty}^{\infty} h(m)x(n-m) \qquad Y(z) = H(z)X(z)$$

Example

- Find the z-transform of the following function $p(n)$



- Write $p(n) = u(n) - u(n-N)$
- Using the shift and linearity properties we obtain

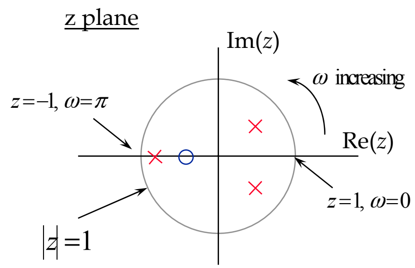
$$P(z) = \frac{1}{1-z^{-1}} - \frac{z^{-N}}{1-z^{-1}} = \frac{1-z^{-N}}{1-z^{-1}}$$

- The ROC is

$$|z^{-1}| < 1$$

Plotting on the z-plane

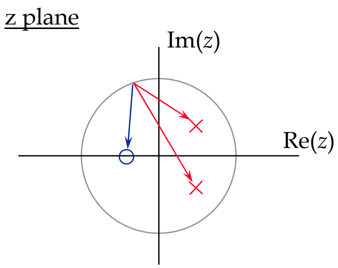
- Given $H(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}$
  - Poles are roots of denominator ×
  - Zeros are roots of numerator ○



Frequency Response from the z-plane plot

- The frequency response is given by $H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$
- Can be derived analytically from the transfer function in z
  - put $z = e^{j\omega}$
- Can be derived graphically
  - compute : $\dfrac{\text{Product of distances to all the zeros}}{\text{Product of distances to all the poles}}$

    as z goes around the unit circle



Inverse z-transform

Aim

- Given $X(z)$ find $x(n)$

4 methods

- Inspection (for power series)
- Long division
- Partial fractions and table look-up
- Inversion formula

Reading: Proakis Section 3.4

Inverse z-transform by inspection

- Given a z-domain expression as a power series

$$X(z) = 1 + 2z^{-1} + 3z^{-2}$$

- use $Z\{A\delta(n-m)\} = Az^{-m}$
- to write

$$x(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2) = \{1, 2, 3\}$$

Inverse z-transform by long division

- Given a z-domain expression as a ratio of polynomials, the first few terms of the sequence can be found by long division.
- Start by converting ratio of polynomials to power series, then use inspection
- E.g.

$$X(z) = \frac{0.5z^2 + 0.5z}{z^2 - z + 0.5}$$

$$\begin{array}{r} 0.5 + 1.0z^{-1} + 0.75z^{-2} + \ldots \\ z^2 - z + 0.5 \;\big)\; 0.5z^2 + 0.5z \qquad\qquad \\ 0.5z^2 - 0.5z + 0.25 \qquad \\ \hline 0 \quad + 1.0z - 0.25 \qquad \\ 1.0z - 1.00 + 0.50z^{-1} \\ \hline 0 \quad + 0.75 - 0.50z^{-1} \end{array}$$

etc

- and hence

$$x(n) = 0.5\delta(n) + 1\delta(n-1) + 0.75\delta(n-2) + \ldots = \{0.5, 1, 0.75, \ldots\}$$

Inverse z-transform by partial fractions and table look-up

- Use tables of standard transform pairs
- Use partial fraction expansion to re-write problem in terms of standard transform pairs
- E.g.

$$X(z) = \frac{4z^2}{z^2 - 0.25}$$

- Use PFE to write

$$X(z) = \frac{2z}{z-0.5} + \frac{2z}{z+0.5}$$

- Use standard transform pair

$$Z\{Aa^n u(n)\} = \frac{Az}{z-a}$$

- to give

$$x(n) = 2(0.5)^n u(n) + 2(-0.5)^n u(n)$$

Inverse z-transform by the inversion formula

- The inverse z-transform is given by

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

- This can be solved using the residue theorem

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz = \sum\left(\text{residues of } X(z)z^{n-1} \text{ at the poles inside contour C}\right)$$

- Express $X(z)z^{n-1}$ as $X(z)z^{n-1} = \frac{\varphi(z)}{(z-z_0)^s}$

  which has s poles at $z = z_0$

- Then $\mathrm{Res}[X(z)z^{n-1} \text{ at } z = z_0] = \frac{1}{(s-1)!}\cdot\left[\frac{d^{s-1}\varphi(z)}{dz^{s-1}}\right]_{z=z_0}$

- Example
  - Find the inverse z-transform of $X(z)=\dfrac{1}{1-az^{-1}}$ for $|z|>|a|$
  - Write $x(n)=\dfrac{1}{2\pi j}\oint_C \dfrac{z^{n-1}}{1-az^{-1}}dz=\dfrac{1}{2\pi j}\oint_C \dfrac{z^n}{z-a}dz$

    C is a circular contour of radius greater than a.
  - Comparing with the form $X(z)z^{n-1}=\dfrac{\varphi(z)}{(z-z_0)^s}$

    gives $s=1$, $z_0=a$ and $\varphi(z)=z^n$.
- For $n \ge 0$ the only pole of $X(z)z^{n-1}$ is at $z=a$ with a residue of a^n

- For $n<0$ there is a multiple order pole at $z=0$
  - For n= -1
    - residue of pole at origin is $-a^{-1}$
    - residue of pole at $z=a$ is a^{-1}

    cancel
  - For n=-2
    - residue of pole at origin is

      $$\mathrm{Res}\left[\frac{1}{z^2(z-a)}\right]_{z=0}=-a^{-2}$$

      residue of pole at $z=a$ is

      $$\mathrm{Res}\left[\frac{1}{z^2(z-a)}\right]_{z=a}=a^{-2}$$

      cancel
  - etc.
- Therefore $x(n)=\begin{cases} a^n & n\ge 0 \\ 0 & n<0 \end{cases}$

Causal and Anticausal Systems

- Already seen that the ROC is the region of the z-plane for which the infinite sum of the z-transform converges
- Given a transfer function $H(z)$ the impulse response $h(n)$ depends on the ROC of $H(z)$

- Causal Example $h(n)=\begin{cases} a^n, & n\ge 0 \\ 0, & n<0 \end{cases} = a^n u(n)$
  - This has z-transform

$$H(z)=\sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

$$=\frac{1}{1-az^{-1}} \quad \text{for } |z|>|a|$$

- Anticausal Example $h(n)=\begin{cases}0, & n\ge 0\\ -a^n, & n<0\end{cases} = -a^n u(-n-1)$

  - This has z-transform

$$H(z)=\sum_{n=-\infty}^{\infty} -a^n u(-n-1)z^{-n} = \sum_{n=-\infty}^{-1}\left(-az^{-1}\right)^n$$
$$= -\frac{1}{1-az^{-1}} \quad \text{for } |z|<|a|$$

- Causal and anticausal sequences have same form of z-transforms but different ROCs

- Generalisation
  - A system with N poles with ROC $|z|>|p_i|$ where p_i is the pole farthest from $z=0$ is causal.
  - A system with N poles with ROC $|z|<|p_i|$ where p_i is the pole nearest to $z=0$ is anticausal.

Stability

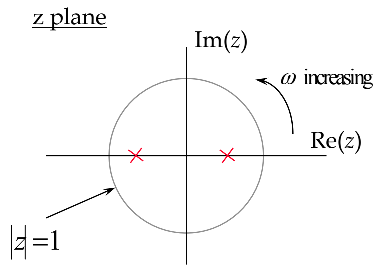
- Two equivalent definitions:
  - A system $H(z)$ is stable if its inverse z-transform $h(n)$ satisfies

$$\sum_{n=-\infty}^{\infty}|h(n)|<\infty$$

  - A system $H(z)$ is stable if its ROC includes the unit circle in the z-plane
- Causal systems are stable if all poles lie inside the unit circle
- Anticausal systems are stable if all poles lie outside the unit circle

- Example

$$H(z)=\frac{1}{(z-0.5)(z+0.75)} \quad \text{for } |z|>0.75$$



  - ROC lies outside both poles
    - therefore system is causal
  - ROC includes unit circle (i.e. modulus of all poles < 1)
    - therefore system is stable

Schur-Cohn Stability Test

- Write the denominator of the system function as

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}$$

- Convert the polynomial coefficients a_k to reflection coefficients K_m
- $A(z)$ has roots within the unit circle iff $|K_m| < 1 \quad \forall m$
- Conversion to reflection coefficients can be done efficiently using a recursive algorithm
  - Levinson/Durbin
  - Uses N^2 multiplications
  - Better than direct factorisation of $A(z)$

- Set $a_N(k) = a_k \quad k = 1,2,\ldots,N$ $\qquad$ N is order of polynomial

  $K_N = a_N(N)$

- Then compute for $m=N, N$-1, ... ,1

$$K_m = a_m(m) \quad a_{m-1}(0) = 1$$

$$b_m(k) = a_m(m-k) \quad k = 0,1,\ldots,m$$

$$a_{m-1}(k) = \frac{a_m(k) - K_m b_m(k)}{1 - K_m^2} \quad k = 1,2,\ldots,m-1$$

Example

$$A(z) = 1 - 1.75z^{-1} - 0.5z^{-2}$$

$$N = 2$$

$$a_2(1) = -1.75, \quad a_2(2) = -0.5$$

$$K_2 = a_2(2) = -0.5$$

$$m = 2:$$

$$K_2 = a_2(2) = -0.5, \quad a_1(0) = 1$$

$$a_1(1) = \frac{a_2(1) - K_2 a_2(1)}{1 - K_2^2} = \frac{-1.75 - 0.5 * 1.75}{1 - 0.25} = -3.5$$

$$m = 1:$$

$$K_1 = a_1(1) = -3.5, \quad a_0(0) = 1$$

Reading: Proakis Chapter 3