Module 2

Discrete Fourier Transform

Contents

- Review of Fourier analysis
  - Fourier series
  - Fourier transform
- Discrete-time Fourier transform
- Discrete Fourier transform
  - Analysis, properties and applications
- Spectral estimation
  - Short-time Fourier transform
  - Windowing
- Fast computation of DFT
  - Decimation-in-time FFT

Review of Fourier Analysis

- Analysis of signals in terms of their components at different frequencies
  - What are signals?
    - Information bearing functions of time
    - usually observed using transducers (microphone, antenna)
    - often represented by analogue voltages or sampled data streams
  - What are frequency components?
    - Fourier analysis considers signals to be constructed from a sum of complex exponentials with appropriate frequencies, amplitudes and phase
    - frequency components are the complex exponentials (sines and cosines) which, when added together, make up the signal

Fourier Series

- For a periodic continuous-time signal $x(t)$ with period T

$$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos k\Omega_0 t + \sum_{k=1}^{\infty} B_k \sin k\Omega_0 t$$

- This can be written in exponential form $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}$
- The a_k are found from $a_k = \frac{1}{T}\int_T x(t)e^{-jk\Omega_0 t}\,dt$
- $x(t)$ is made from a d.c. term plus a weighted sum of sinusoids at integer multiples of the fundamental frequency $\Omega_0 = 2\pi/T$

Fourier Transform

- For an aperiodic signal, we can say that it is equivalent to a periodic signal with infinite period
  - hence the Fourier Series becomes the Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

Existence of Fourier Transform

- A (Dirichlet Conditions) $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
  - $x(t)$ is absolutely integrable
  - $x(t)$ has a finite number of maxima and minima in any finite interval
  - $x(t)$ has a finite number of discontinuities within any finite interval and all discontinuities are finite
- B
  - If $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$ then

    1. $X(\Omega)$ is finite

    2. $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

    for $e(t) = \hat{x}(t) - x(t)$
    where $\hat{x}(t) = IFT\{X(\omega)\}$

Discrete-Time Signals and the Fourier Transform

- So far, so good - mostly review
- Now consider a discrete-time signal *x*(*n*) and its z-transform *X*(*z*)
  - *x*(*n*) is just a sequence of numbers
  - what does it mean for a sequence of numbers to have a spectrum?
- How can the spectrum of a discrete-time signal (sequence) be found using the Fourier transform when the Fourier transform is defined for continuous-time signals?

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Discrete-Time Fourier Transform

- Consider a discrete-time signal *x*(*n*) and a continuous-time signal *x'*(*t*) given by *

$$x'(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT)$$

- Now

$$X'(e^{j\omega}) = \int_{-\infty}^{\infty} x'(t)e^{-j\omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)\int_{-\infty}^{\infty} e^{-j\omega t}\delta(t - nT)dt$$

$$= \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega T}$$

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- Hence
  - If $x(n)$ is an infinite sequence, we can compute the discrete-time Fourier transform (DTFT)
  - a sequence of numbers having a spectrum is OK if we use the trick of *
  - the DTFT is a continuous function of frequency

Inverse DTFT

- There exists and inverse DTFT

$$x(n) = \frac{T}{2\pi} \int_{\frac{2\pi}{T}} X'(e^{j\Omega}) e^{jn\omega T} \, d\omega$$

  - where $\int_{\frac{2\pi}{T}}$ means integrate over any $\frac{2\pi}{T}$ range of ω

Discrete Fourier Transform

- The DTFT is defined using an infinite sum over a discrete-time signal and yields a continuous function
  - not very useful since we don't often have real signals of infinite length
- Now introduce the DFT which is defined using a sum over only N (finite) samples of a discrete-time signal
- The DFT yields a discrete function of frequency with points uniformly spaced around the unit circle in z
  - The DFT yields a sampled version of the DTFT

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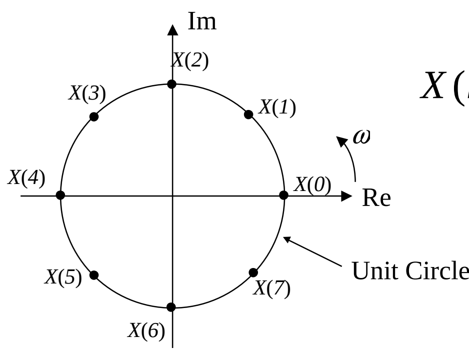

$$X(k) = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

- DTFT yields a continuous function
  - $X(\omega)$ is defined at all values of z around the unit circle
- DFT yields sample version
  - $X(\omega)$ is defined only at certain points around the unit circle

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Applications of the DFT

- Spectrum analysis of periodic discrete-time signals
- Short-time spectral estimation of aperiodic signals
  - normally in conjunction with a window
- Fast implementation of convolution (filtering) via fast algorithm for DFT (FFT).

Derivation of DFT from DTFT

- DTFT: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega T}$
  - Consider only *N* samples of *x*(*n*), *n*=0, 1, … *N*-1 and compute only *N* samples of $X(e^{j\omega})$ using

$$\omega = k\omega_0 \qquad k = 0,1,\ldots,N-1 \qquad \omega_0 = 2\pi/NT$$

Sampling frequency divided by *N*

  - giving

$$X(e^{jk\omega_0}) = \sum_{n=0}^{N-1} x(n)e^{-jkn\omega_0 T}$$

  - which can be written

$$X(k) = \sum_{n=0}^{N-1} x(n)W^{kn} \qquad k = 0,1,\ldots,N-1 \qquad W = e^{-j2\pi/N}$$

Inverse DFT

- There exists an inverse relationship

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-kn} \qquad n = 0,1,\ldots,N\text{-}1$$

Notes and Summary

- Notes
  - $X(k)$ is the DFT of $x(n)$
  - $X(k)$ is an N-point sequence computed from $x(n)$, another N-point sequence
  - k and n are dimensionless variables
  - k is the frequency index, n is the time index
- Summary

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0,1,\ldots,N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0,1,\ldots,N-1$$

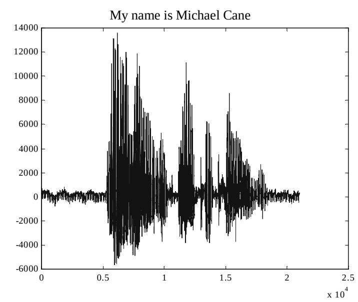
Properties of the DFT

- Linearity: $Ax_1(n) + Bx_2(n) \leftrightarrow AX_1(k) + BX_2(k)$
  - *A*, *B*, arbitrary constants
- Delay: $x(n-m)_{\mathrm{mod}\,N} \leftrightarrow X(k)W^{km}$
  - where $x(q)_{\mathrm{mod}\,N} = x(q \pm iN)$ i integer
- Modulation: $W^{-nl}x(n) \leftrightarrow X(k-l)_{\mathrm{mod}\,N}$
  - Modulation in one domain implies circular shift in the other

- Convolution: $\sum_{m=0}^{N-1} x_1(m)x_2(n-m)_{\mathrm{mod}\,N} \leftrightarrow X_1(k)X_2(k)$
  - circular convolution of two sequences in the product of their DFTs
- Multiplication: $x_1(n)x_2(n) \leftrightarrow \frac{1}{N}\sum_{l=0}^{N-1} X_1(l)X_2(k-l)_{\mathrm{mod}\,N}$
  - multiplication in one domain implies circular convolution in the other
- Conjugation: $x^*(n) \leftrightarrow X^*(-k)_{\mathrm{mod}\,N}$

Spectral Estimation using DFT

- Consider a speech signal $s(t)$:
  - amplitude spectrum is controlled by the voice source (vocal cords) and the articulators (lips, tongue etc)
- Suppose that:
  - $s[n]$ is a sampled version of $s(t)$ at sampling frequency f_s Hz,
- Question:
  - find the amplitude and phase spectra at time t_1. Assume the spectrum of $s(t)$ is constant near t_1.



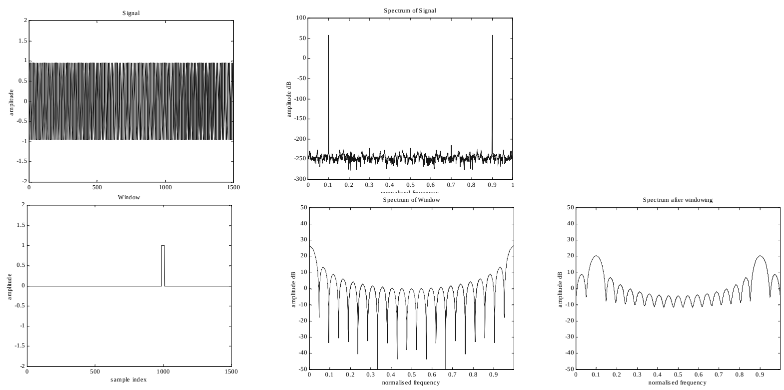
1. Windowing

- we must select N samples near t_1.
- use a windowing function $w(n)$
  - eg: rectangular, hanning, hamming etc
- create a new data record $s_1(n) = s(m).w(n)$
  - N is the size of the window
  - position the window in the region of t_1 and multiply the signal by the window. The result is a new data record $s_1(n)$ of length N.
  - N is chosen to be an integer power of 2 so that the DFT can be implemented using a fast algorithm (FFT).

$$n = 0, 1, \ldots, N-1$$

$$m = m_1 - \frac{N}{2}, \ldots, m_1 + \frac{N}{2} - 1$$

- from the properties of the DFT
  - multiplying the signal by a window is equivalent to convolving their Fourier transforms
  - using a rectangular window causes convolution with a $\frac{\sin(x)}{x}$
  - the errors caused by the convolution are called leakage



- the rectangular window produces relatively narrow main lobes with high amplitude side-lobes (~ -13dB)
- the rectangular window introduces no leakage in the special case where $x(n)$ is periodic with period N
- other windows produce broader main lobes with lower amplitude side-lobes, eg:
  - Hamming $\quad w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$
  - Hanning $\quad w(n) = 0.50 - 0.50\cos\left(\frac{2\pi n}{N-1}\right)$

$$0 \le n \le N-1$$

2. Discete Fourier Transform

- Given $x_1(n) = x(m) \cdot w(n)$

  compute $X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}$

3. Interpret the result

- what is $X_1(k)$?
- $X_1(k)$ is a sampled version of the discrete-time Fourier transform of $x_1(n)$
- the samples in the frequency domain are spaced by Δf

$$\Delta f = \frac{f_s}{N} = \frac{1}{NT} = \frac{1}{\text{Total time of sampling}}$$

- $X_1(0)$ is the d.c. component
- $X_1(N/2)$ is the component at the Nyquist frequency
- $X_1(N\text{-}1)$ is the component at $f_s - \Delta f$

4. Present the result

- normally we graph the result of a spectrum estimate
- several options:
  - power spectrum (spectrogram) *S*(*k*)
    - tells you how much power there is in the signal at each discrete frequency

$$S(k) = \tfrac{1}{N}\left|X(k)\right|^2 = \tfrac{1}{N} X^*(k).X(k) \qquad k = 0, 1, \ldots, N-1$$

  - Amplitude and phase spectrum
    - define the amplitude and phase spectra to be

$$\left|X(k)\right| = \sqrt{\left(\mathrm{Re}\{X(k)\}\right)^2 + \left(\mathrm{Im}\{X(k)\}\right)^2}$$

$$\angle X(k) = \tan^{-1}\left(\frac{\mathrm{Im}\{X(k)\}}{\mathrm{Re}\{X(k)\}}\right)$$