Imperial College London

Module 5 - Multirate Signal Processing

Patrick A. Naylor

Contents

- Applications of multirate signal processing
- Fundamentals
  - decimation
  - interpolation
- Resampling by rational fractions
- Multirate identities
- Polyphase representations
- Maximally decimated filter banks
  - aliasing
  - amplitude and phase distortion
  - perfect reconstruction conditions

Introduction

- In single-rate DSP systems, all data is sampled at the same rate
  - no change of rate within the system.
- In multirate DSP systems, sample rates are changed (or are different) within the system
- Multirate can offer several advantages
  - reduced computational complexity
  - reduced transmission data rate.

Example: Audio sample rate conversion

- recording studios use 192 kHz
- CD uses 44.1 kHz
- wideband speech coding using 16 kHz

- master from studio must be rate-converted by a factor

$$\frac{44.1}{192}$$

Example: Oversampling ADC

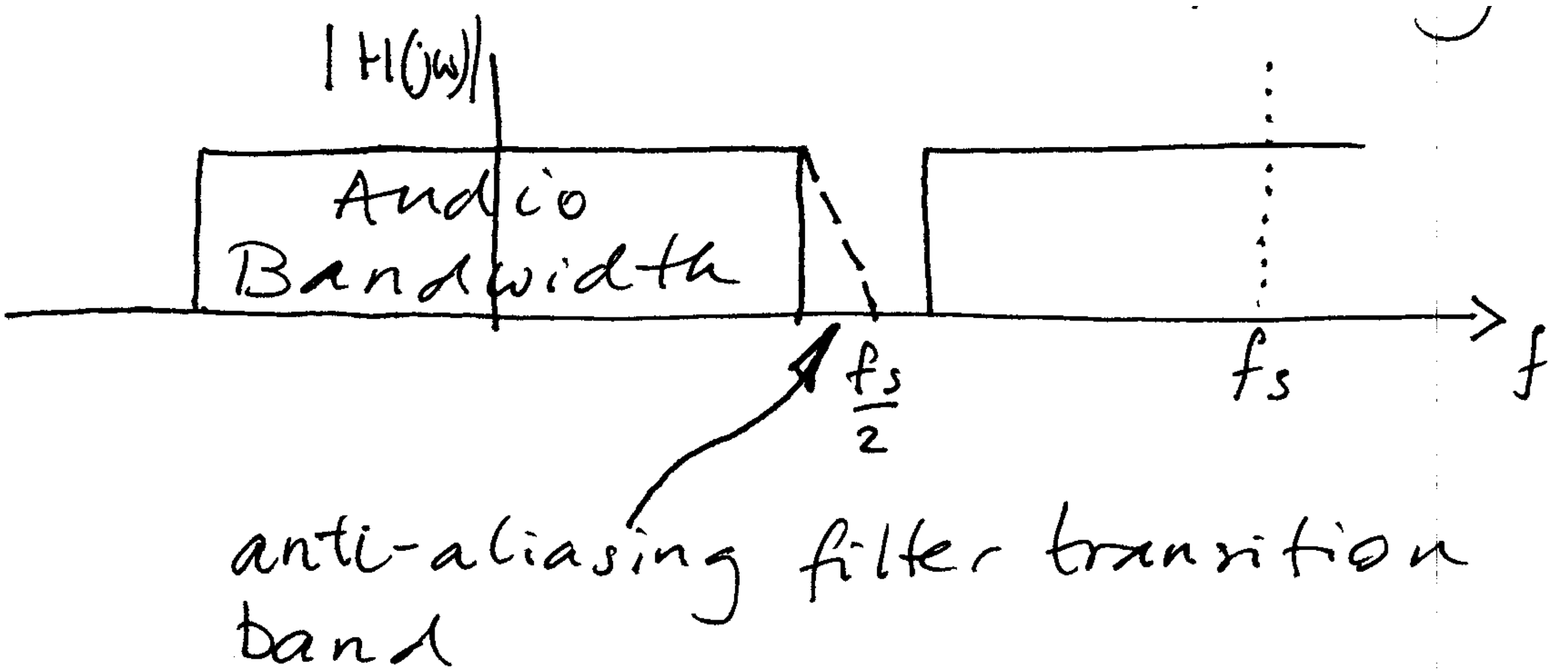
Consider a Nyquist rate ADC in which the signal is sampled at the desired precision and at a rate such that Nyquist's sampling criterion is just satisfied.

- Bandwidth for audio is 20 Hz $< f <$ 20 kHz
- Antialiasing filter required has very demanding specification

$$|H(j\omega)| = 0 \text{ dB}, \; f < 20 \text{ kHz}$$
$$|H(j\omega)| < 96 \text{ dB}, \; f \geq \frac{44.1}{2} \text{ kHz}$$

- Requires high order analogue filter such as elliptic filters that have very nonlinear phase characteristics
  - hard to design, expensive and bad for audio quality.

Nyquist Rate Conversion Anti-aliasing Filter.



Consider oversampling the signal at, say, 64 times the Nyquist rate but with lower precision. Then use multirate techniques to convert sample rate back to 44.1 kHz with full precision.

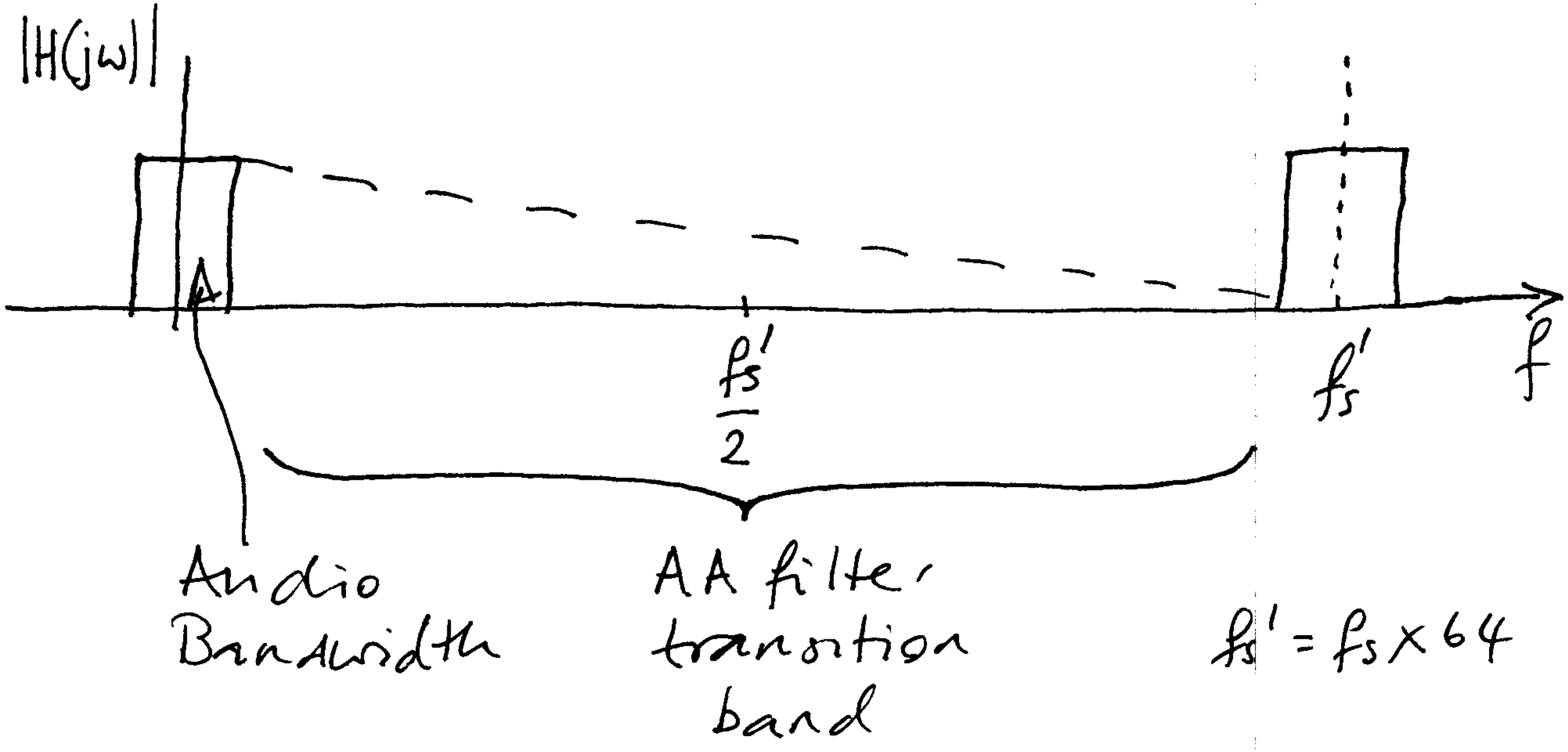
- New (over-sampled) sampling rate is 44.1 $\times$ 64 kHz.
- Requires simple antialiasing filter

$$|H(j\omega)| = 0 \text{ dB}, \; f < 20 \text{ kHz}$$

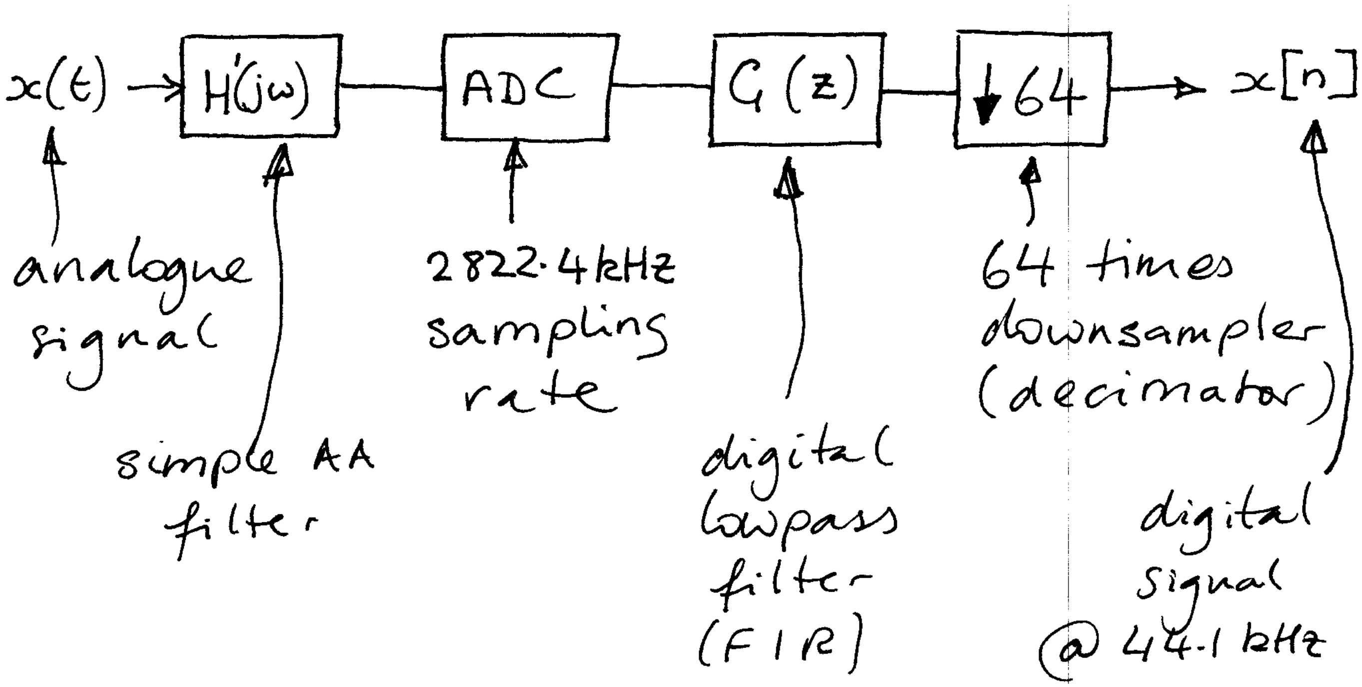
$$|H(j\omega)| < 96 \text{ dB}, \; f \geq (44.1 \times 64) - \frac{44.1}{2} \text{ kHz}$$

- Could be implemented by simple filter (eg. RC network)
- Recover desired sampling rate by downsampling process.

Oversampled Conversion Antialiasing Filter



Overall System



- This is a simplified version
- In these lectures we will study blocks like $G(z)$ and $\downarrow 64$

Example: Subband Coding

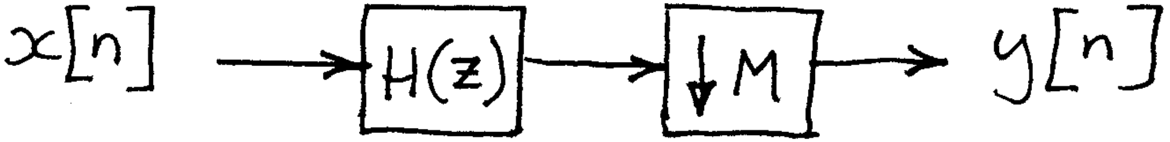
Consider quantizing the samples of a speech signal. How many bits are required?

- In general, 16 bits precision per sample is normally used for audio. This gives an adequate dynamic range.
- In practice, certain frequency bands are less important perceptually because they contain less significant information
  - bands with less information or lower perceptual importance may be quantized with lower precision - fewer bits.
- Divide the spectrum of the signal into several subbands then allocate bits to each band appropriately.

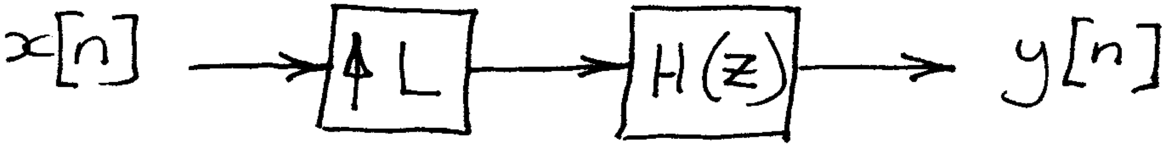
- 16 bits per sample, 10 kHz sampling frequency gives
  - 160 kbits/s
- Divide into 2 bands: high frequency and low frequency subbands.
  - High frequencies of speech are less important to intelligibility.
  - Therefore use only 8 bits per sample
- The sampling frequency can be reduced by a factor of 2 since bandwidth is halved, still satisfying Nyquist criterion.
  - $5 \times 16 + 5 \times 8 = 120$ kbits/s
  - 4:3 compression
- Reconstructed signal has no noticeable reduction is signal quality.

Fundamental Multirate Operations

- Downsampling by a factor M
  - filter and M-fold decimator



- Upsampling by a factor L
  - L-fold expander and filter



M-fold Decimator

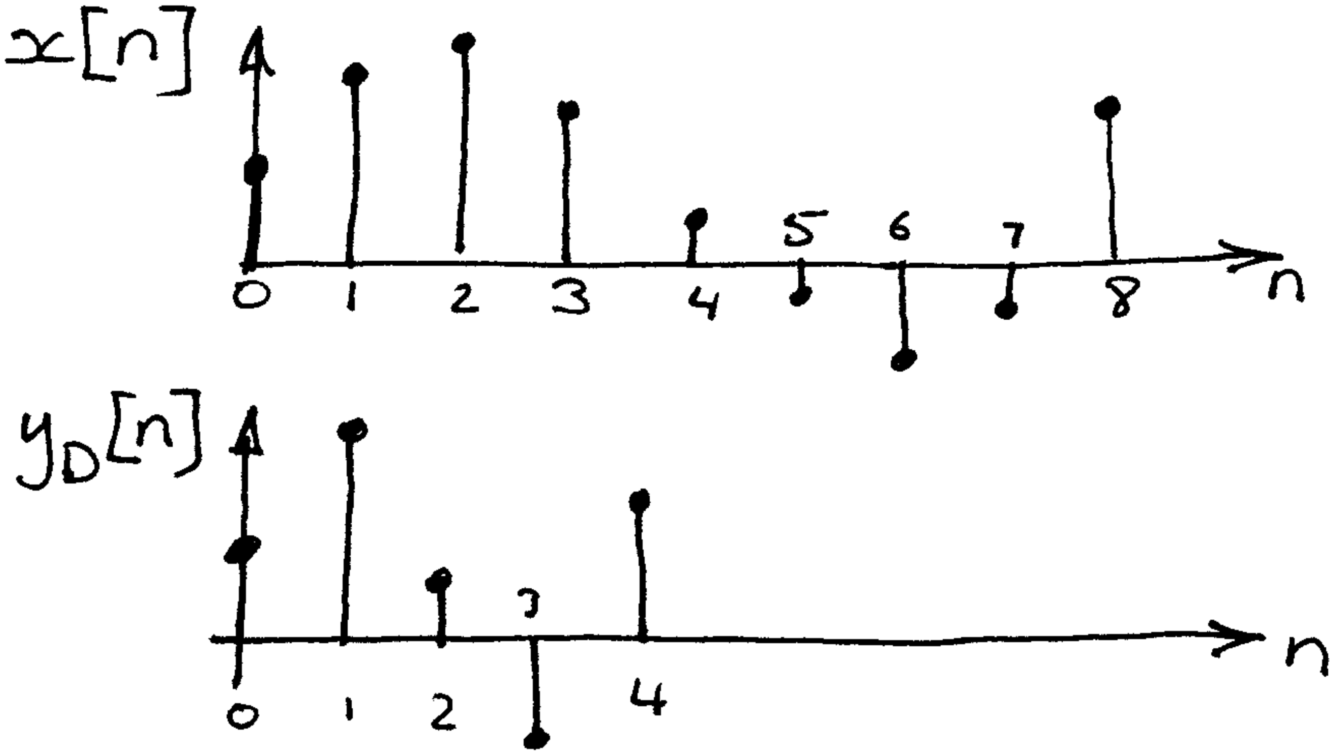
For an input sequence $x(n)$, select only the samples which occur at integer multiples of M. The other samples are thrown away.

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y_D[n]$$

$$y_D(n) = x(Mn)$$

- Aliasing will occur in $y_D(n)$ unless $x(n)$ is sufficiently bandlimited
  - loss of information.

- Eg. $M = 2$



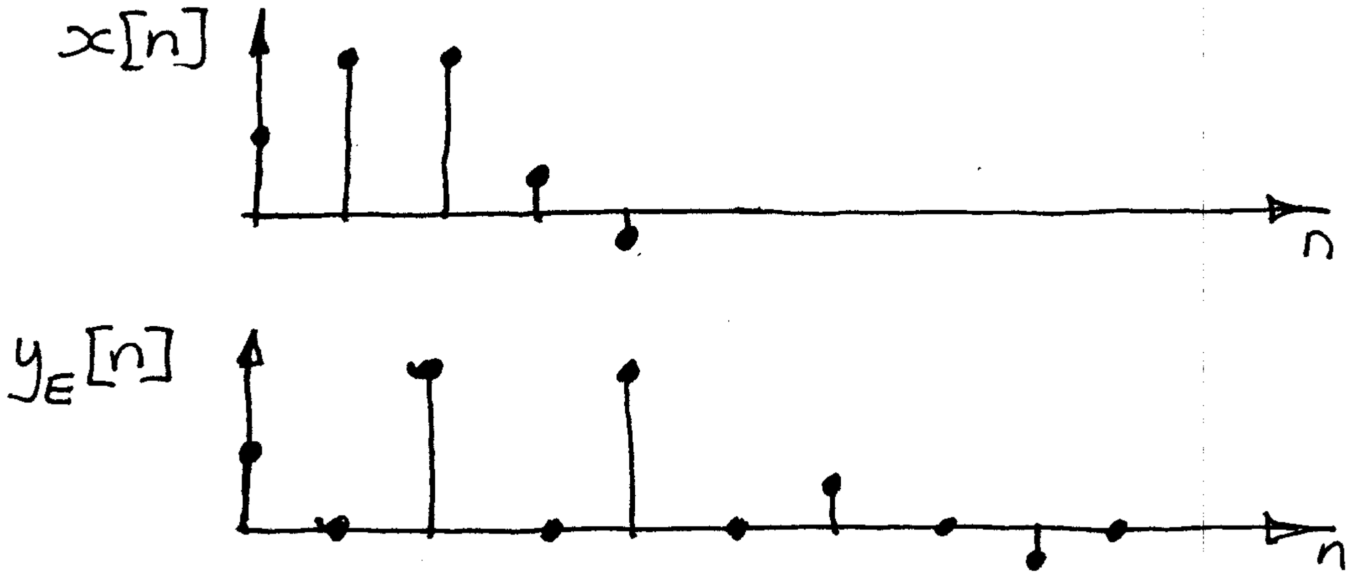
L-fold Expander

For an input sequence $x(n)$, insert $L-1$ zeros between each sample.

$$y_E(n) = x(Mn)$$

- $x(n)$ can always be recovered from $y_E(n)$
  - no loss of information, no aliasing.

- Eg. $L = 2$



Frequency Domain View of the Expander

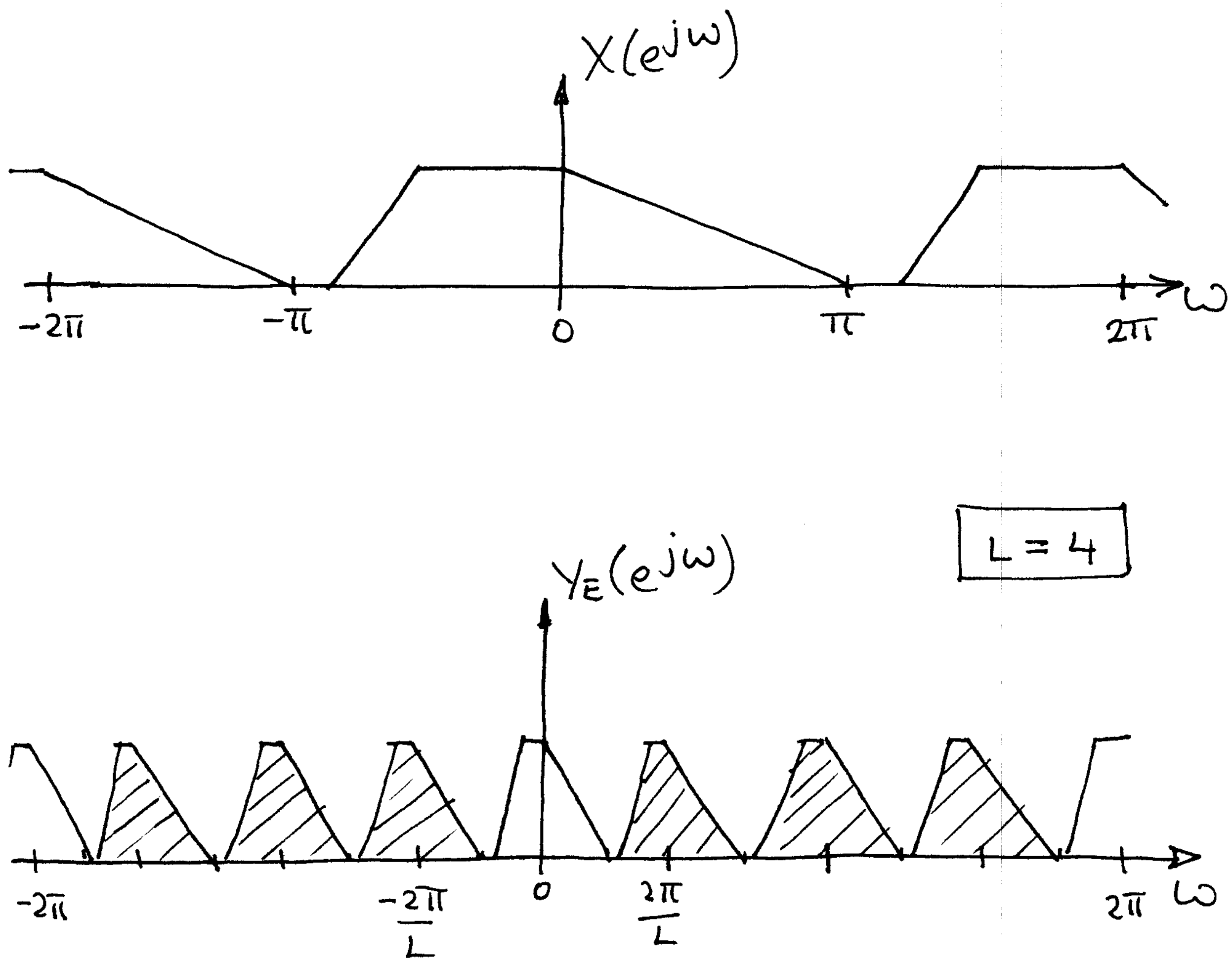
From the definition of the z-transform

$$
\begin{aligned}
Y_E(z) &= \sum_{n=-\infty}^{\infty} y_E(n) z^{-n} \\
&= \sum_{k=-\infty}^{\infty} y_E(kL) z^{-kL} \\
&= \sum_{k=-\infty}^{\infty} x(k) z^{-kL} = X(z^L)
\end{aligned}
$$

For frequency response write $z = e^{j\omega}$ giving

$$Y_E(e^{j\omega}) = X(e^{j\omega L})$$

- Y_E is a compressed version of X
- Multiple images of $X(e^{j\omega})$ are created in $Y_E(e^{j\omega})$ between $\omega = 0$ and $\omega = 2\pi$



- To use the expander for interpolation, a lowpass filter is applied after the expander to remove the images (shaded).

Frequency Domain View of the Decimator

From the definition of the z-transform

$$Y_D(z) = \sum_{n=-\infty}^{\infty} y_D(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(Mn) z^{-n}$$

Let

$$x_1(n) = \begin{cases} x(n) & \text{if } n \text{ is an integer multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

Then

$$Y_D(z) = \sum_{n=-\infty}^{\infty} x_1(Mn) z^{-n} = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k/M}$$

since $x_1(k) = 0$ unless k is a multiple of M.

Therefore

$$Y_D(z) = X_1(z^{1/M}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$$

as will be shown on the next slide and using

$$W_M^k = e^{-j2\pi k/M}$$

For frequency response write $z = e^{j\omega}$ to give

$$Y_D(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega - 2\pi k)/M}\right)$$

We arrive at the previous expression for $Y_D(z)$ by considering a new sequence

$$c_M(n) = \begin{cases} 1 & \text{if } n \text{ is an integer multiple of } M \\ 0 & \text{otherwise} \end{cases}$$

and then writing

$$x_1(n) = c_M(n)x(n)$$

Further consideration of $c_M(n)$ tells us that $c_M(n)$ is the inverse Fourier transform of unity and can be written

$$c_M(n) = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{-kn}$$

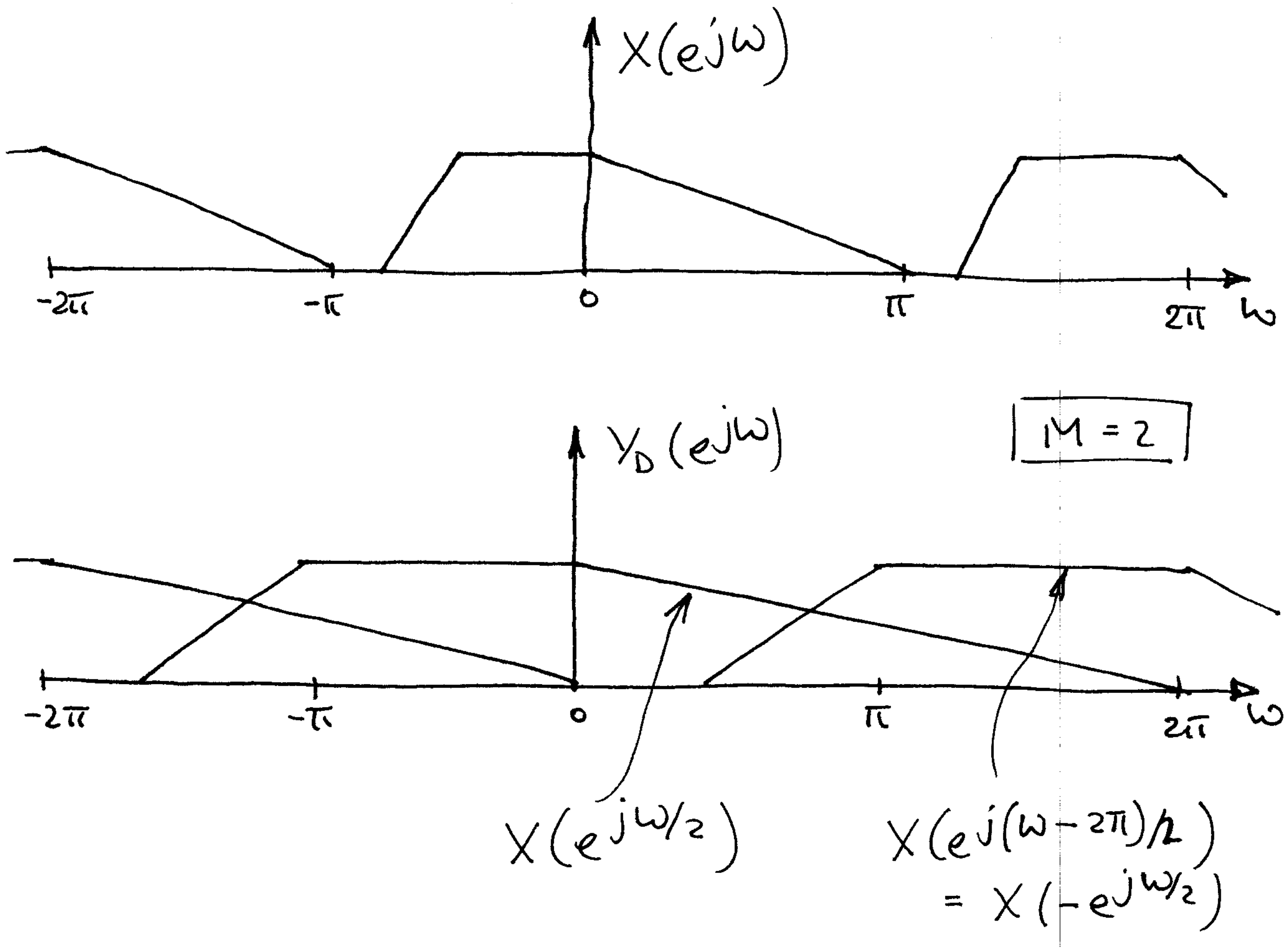
Then

$$
\begin{aligned}
X_1(z) &= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)W_M^{-kn}z^{-n} \\
&= \frac{1}{M}\sum_{k=0}^{M-1}\sum_{n=-\infty}^{\infty} x(n)\left(W_M^k z\right)^{-n} \\
&= \frac{1}{M}\sum_{k=0}^{M-1} X\left(zW_M^k\right)
\end{aligned}
$$

from the definition of the z-transform. So finally

$$
Y_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M}W_M^k\right)
$$

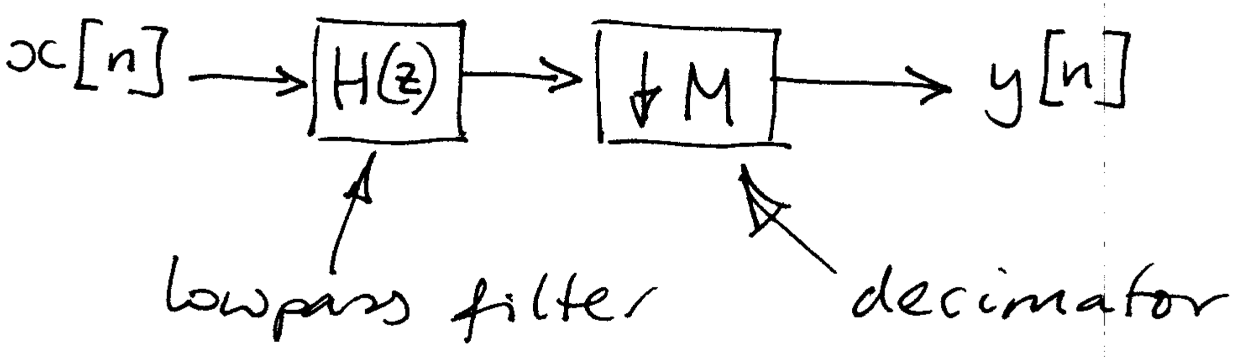
- What does $Y_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{1/M} W_M^k\right)$ represent?
  - stretching of $X(e^{j\omega})$ to $X(e^{j\omega/M})$
  - creating $M-1$ copies of the stretched versions
  - shifting each copy by successive multiples of 2π and superimposing (adding) all the shifted copies
  - dividing the result by M



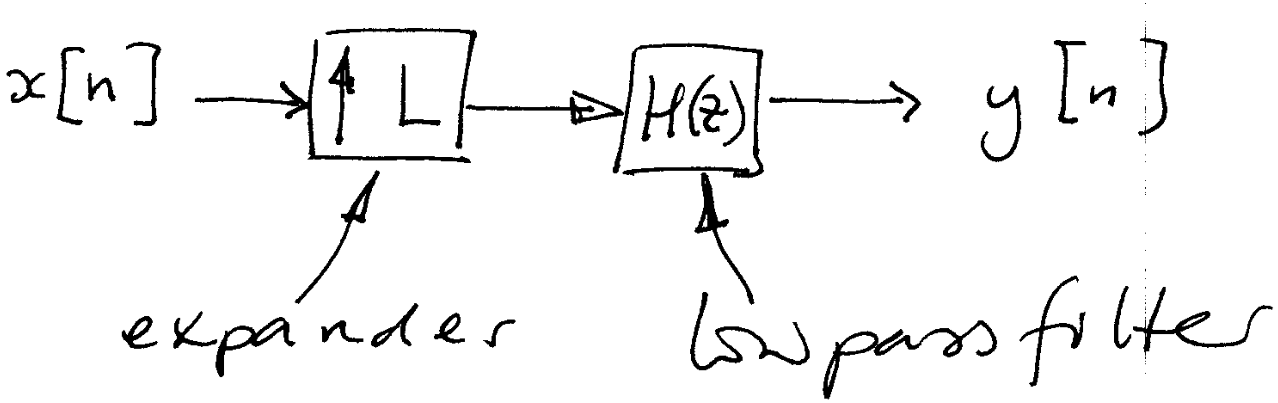
- To use a decimation process we must first bandlimit the signal to $|\omega| < \frac{\pi}{M}$.

Summary

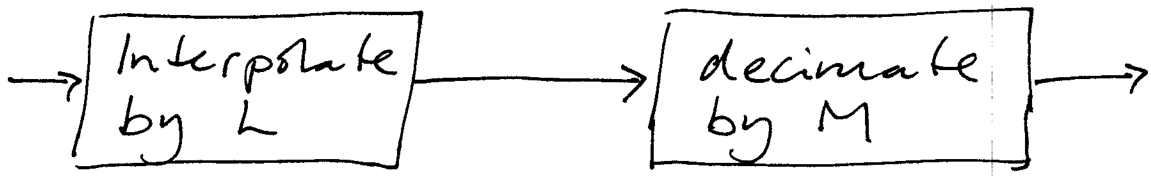
- Downsampling

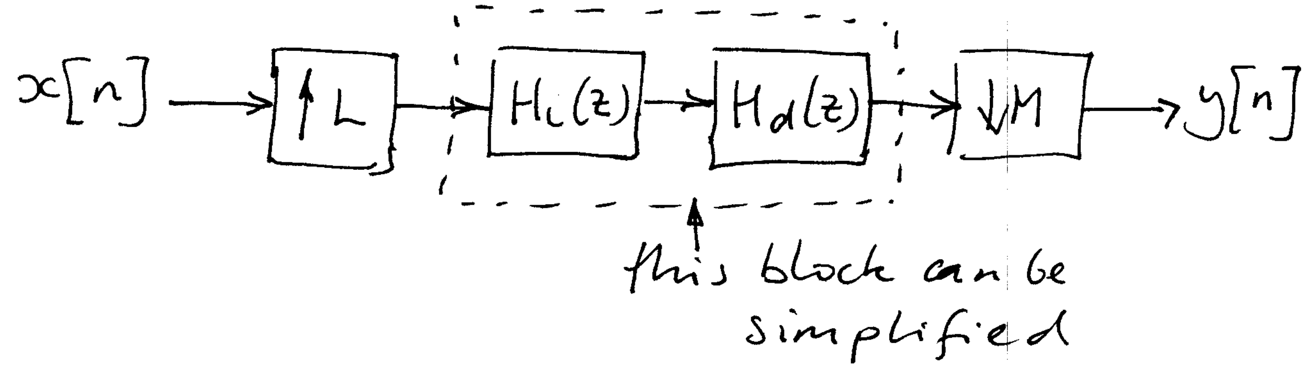


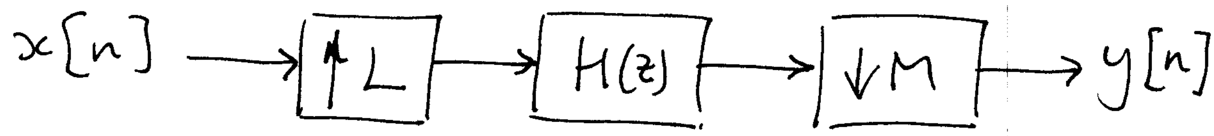
- Upsampling



Resampling by a Rational Fraction L/M







H(z) is designed to attenuate the images of x[n] created by the expander

If $L > M$, the system will not introduce aliasing (H(z) designed as stated)

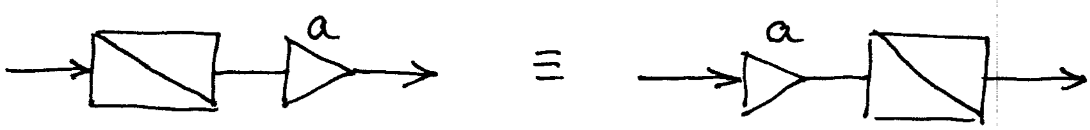
If $M > L$, x[n] must be bandlimited to the "new" nyquist rate - either intrinsically or by H(z)

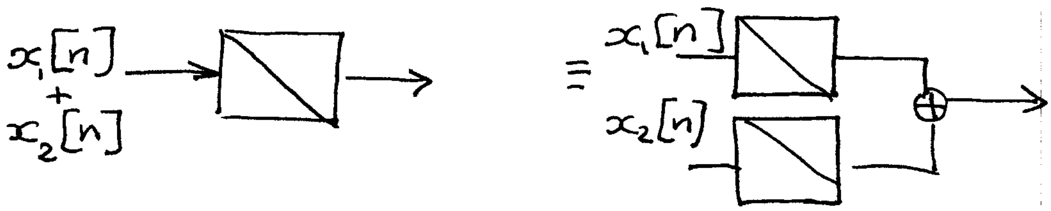
Sufficient conditions on H(z)

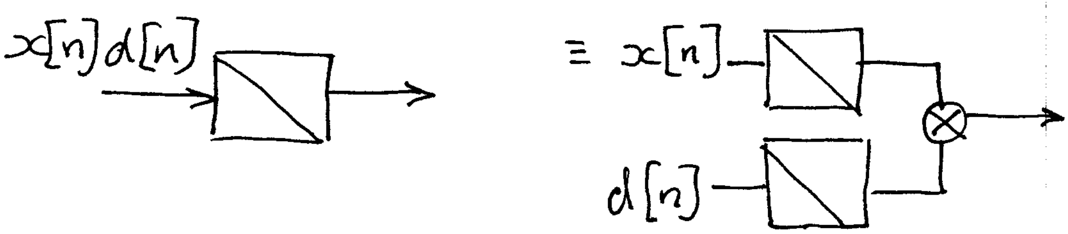
$$|H(e^{j\omega})| = 0, \quad \omega \geq \frac{\omega_{sn}}{2}$$

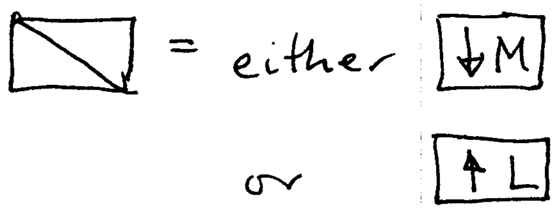
new sampling freq.

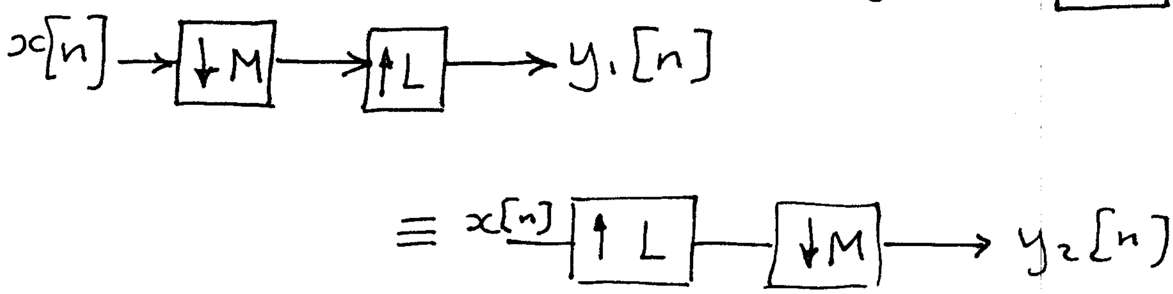
Some Multirate Identities











iff L and M are relative prime

(greatest common divisor = 1)

The Noble Identities

$$x[n] \to \boxed{\downarrow M} \to \boxed{G(z)} \to y_1[n] \quad \equiv \quad x[n] \to \boxed{G(z^M)} \xrightarrow{x_2[n]} \boxed{\downarrow M} \to y_2[n]$$

$$x[n] \to \boxed{G(z)} \to \boxed{\uparrow L} \to y_3[n] \quad \equiv \quad x[n] \to \boxed{\uparrow L} \to \boxed{G(z^L)} \to y_4[n]$$

since

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W^k)\, G(z)$$

$$X_2(z) = X(z)\, G(z^M)$$

$$Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{1/M} W^k)\, G((z^{1/M} W^k)^M)$$

$$= Y_1(z)$$

$\left[\; W^{kM} = e^{-j2\pi k} = 1 \right.$ for integer $\left. k \;\right]$

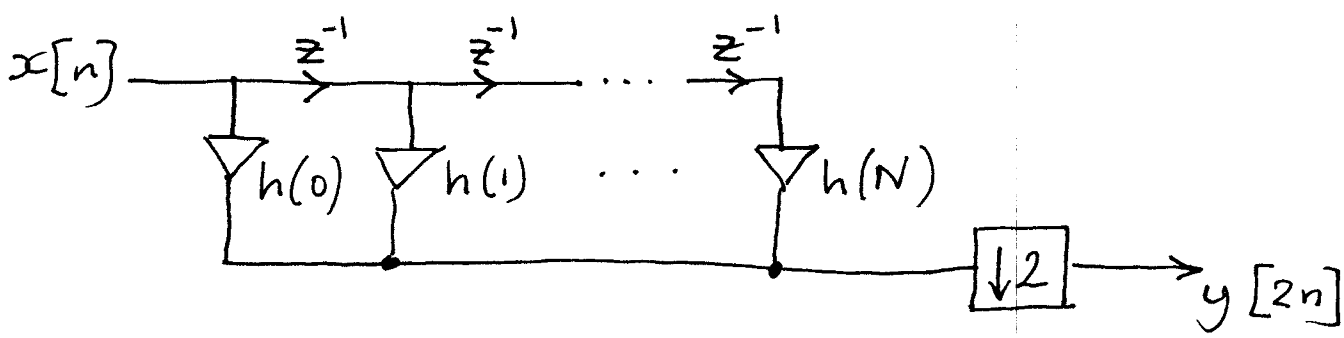
Similarly for $Y_3(z)$ and $Y_4(z)$

Polyphase Representation of filters

Motivation:

- efficient realisation of filters for multirate DSP.

Consider a classical FIR filter followed by a down-by-two decimator



- the output only produces a sample for even samples (n even)

- the computations of the odd samples are "thrown away" by the decimator.

- polyphase structures avoid this waste.

Consider the FIR example

$$\frac{Y(z)}{X(z)} = H(z) = 1 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

$$y[n] = x[n] + h(1)x[n-1] + h(2)x[n-2] + h(3)x[n-3]$$

$$y[n+1] = x[n+1] + h(1)x[n] + h(2)x[n-1] + h(3)x[n-2]$$

$$y[n+2] = x[n+2] + h(1)x[n+1] + h(2)x[n] + h(3)x[n-1]$$

$$y[n+3] = x[n+3] + h(1)x[n+2] + h(2)x[n+1] + h(3)x[n]$$

$$y[n+4] = x[n+4] + h(1)x[n+3] + h(2)x[n+2] + h(3)x[n+1]$$

$$\vdots$$

$$y[\text{even}] = \sum x[\text{even}] \cdot h(\text{even}) + \sum x[\text{odd}] \cdot h(\text{odd})$$

$$y[\text{odd}] = \sum x[\text{odd}] \cdot h(\text{even}) + \sum x[\text{even}] \cdot h(\text{odd})$$

$y[\text{odd}]$ terms are not needed because of $\boxed{\downarrow 2}$

$\therefore$

$$\therefore\ y[2n] = \sum_k x[2(n-k)]\,h(2k) + \sum_k x[2(n-k)-1]\,h(2k+1)$$

even outputs

even delay inputs convolved with even taps

odd delay inputs convolved with odd taps

$$\therefore\ Y(z) = \sum_k X(z)\,z^{-2k}\,h(2k) + \sum_k X(z)\,z^{-(2k+1)}\,h(2k+1)$$

where $Y(z) = \mathcal{Z}\{y[2n]\}$

$X(z) = \mathcal{Z}\{x[2n]\}$

$$\therefore\ H_p(z) = \frac{Y(z)}{X(z)} = \sum_k h(2k)\,z^{-2k} + z^{-1}\sum_k h(2k+1)\,z^{-2k}$$

$$\equiv E_0(z^2) + z^{-1}\,E_1(z^2)$$

This is the two-phase decomposition of $H(z)$.

Summary and Generalisation

Two Phases

For any filter $H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$

we can write

$$H(z) = \sum_{k=-\infty}^{\infty} h(2k) z^{-2k} + z^{-1} \sum_{k=-\infty}^{\infty} h(2k+1) z^{-2k}$$

$$= E_0(z^2) + z^{-1} E_1(z^2)$$

for $E_0(z) = \sum_{k=-\infty}^{\infty} h(2k) z^{-k}$

$$E_1(z) = \sum_{k=-\infty}^{\infty} h(2k+1) z^{-k}$$

M-phases

$$H(z) = \sum_{k=-\infty}^{\infty} h(kM)\, z^{-kM}$$

$$+ z^{-1} \sum_{k=-\infty}^{\infty} h(kM+1)\, z^{-kM}$$

$$\vdots$$

$$+ z^{-(M-1)} \sum_{k=-\infty}^{\infty} h(kM+M-1)\, z^{-M}$$

or more compactly:

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M)$$

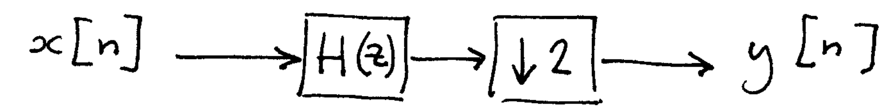
$$E_l(z) = \sum_{k=-\infty}^{\infty} e_l(k)\, z^{-k}$$

$$e_l(n) = h(kM+l) \qquad 0 \leqslant l \leqslant M-1$$

- called Type I polyphase representation.

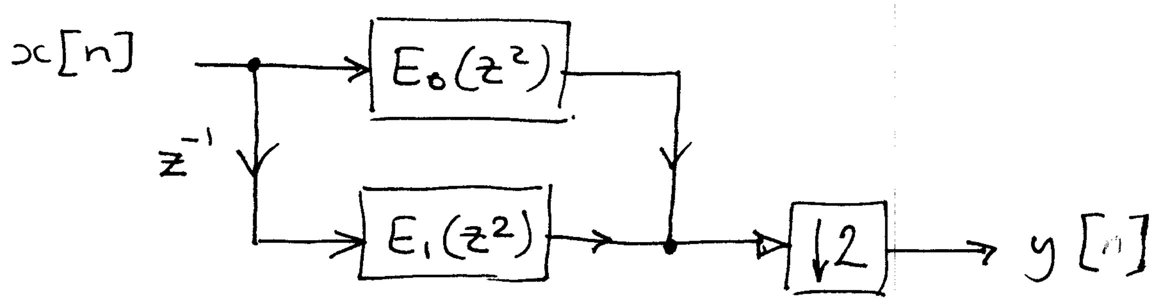
Efficient Polyphase Structures

Consider the decimation filter for $M = 2$

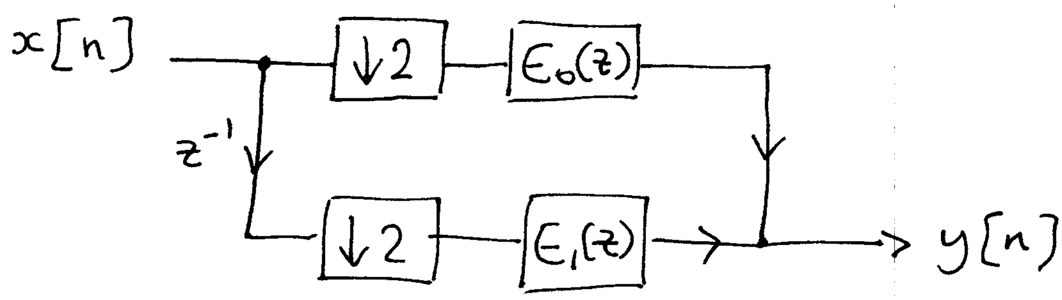


In polyphase representation

$$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$$



From the Noble Identities we obtain



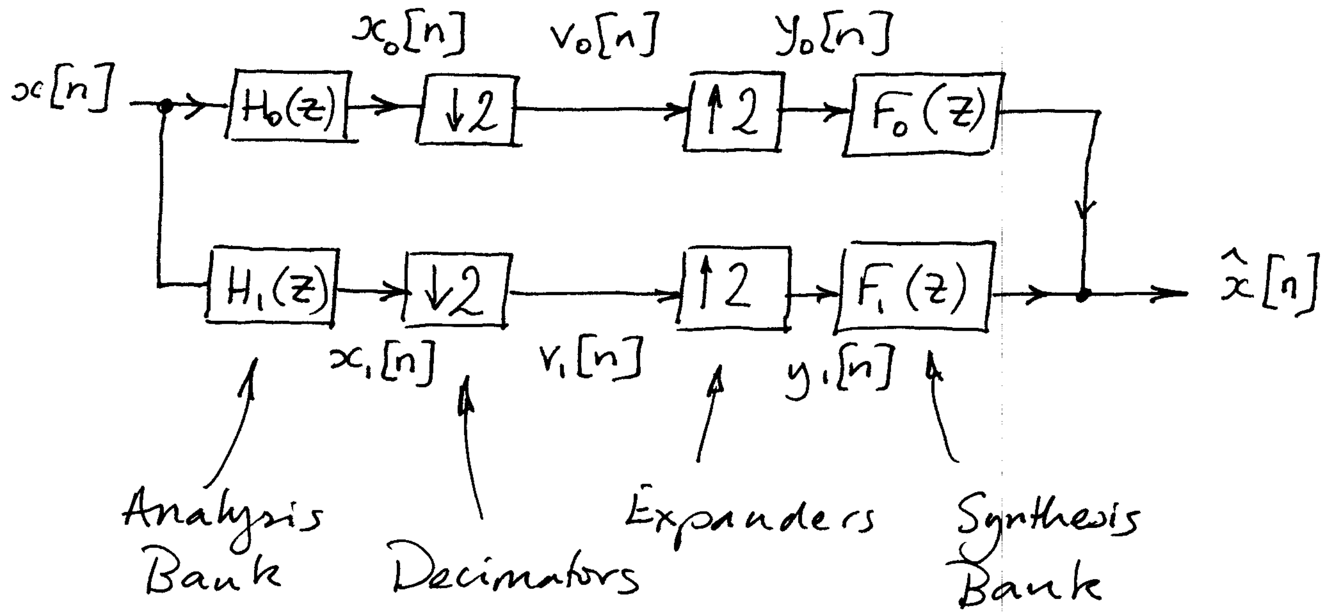
- the filtering is performed at the lower rate.

Maximally Decimated Filter Banks

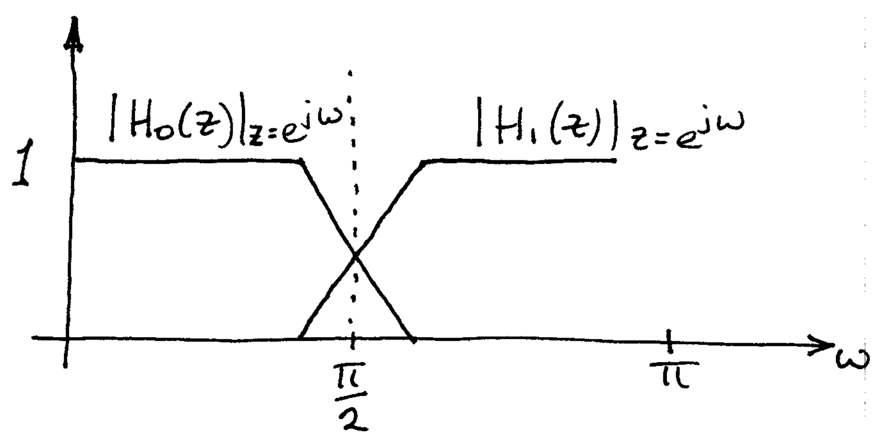
- divide a signal into M subbands
  - analysis bank
- reconstruct a signal from M subband signals
  - synthesis bank
- allows processing of each subband separately
  - coding
  - filtering
    etc.

Here we focus on the filterbanks, not the processing.

Example : Two channel maximally decimated filter bank



Normally, processing is applied to v_0 and v_1
– ignore for purpose of analysis.



– Extension to M bands : $H(z)$ bandpass.

When $|H_0(z)|$ and $|H_1(z)|$ are mirror images around $\pi/2$, they are said to be QMF (Quadrature Mirror Filters)

$H_1(z)$ can be designed to mirror $H_0(z)$ by setting

$$H_1(z) = H_0(-z)$$

i.e. $$h_1[n] = h_0[n] \cdot (-1)^n$$

where $h[n]$ is the impulse response of $H(z)$.

Errors introduced by the filter banks.

In a perfect reconstruction system:

$$\hat{x}[n] = c\,x[n-n_0]$$

- reconstruction is exactly to within a multiplicative constant and a delay.

Otherwise, 3 possible types of distortion:

- aliasing
  - $x_0[n]$ and $x_1[n]$ are not sufficiently bandlimited
- amplitude distortion

$$|T(e^{j\omega})| = \left| \frac{\hat{X}(z)}{X(z)} \right|_{z=e^{j\omega}} \neq C$$

- phase distortion (non-linear phase)

$$\angle T(e^{j\omega}) = \tan^{-1}\left(\frac{Im\left\{\frac{\hat{X}(z)}{X(z)}\right\}_{z=e^{j\omega}}}{Re\left\{\frac{\hat{X}(z)}{X(z)}\right\}_{z=e^{j\omega}}} \right) \neq a + b\omega$$

[a, b are constants.]

Analysis of the 2 band case

Review:

For the decimators, we have

$$Y_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{1/M} W^{k}\right) \qquad x[n] \rightarrow \boxed{\downarrow M} \rightarrow y_D[n]$$

For $M = 2$

$$Y_D(z) = \frac{1}{2}\left(X(z^{1/2}) + X(-z^{1/2}) \right)$$

$$\left[\, W = e^{-j2\pi/M} \,\right]$$

desired component $X(e^{jW/2})$

aliasing component $X(e^{j(\pi - W)/2})$

For the expander, we have

$$Y_E(z) = X(z^{L}) \qquad x[n] \rightarrow \boxed{\uparrow L} \rightarrow y_E[n]$$

Using these relationships:

$$X_k(z) = H_k(z)\,X(z) \qquad k = 0, 1$$

$$V_k(z) = \frac{1}{2}\left(X_k(z^{1/2}) + X_k(-z^{1/2})\right) \qquad k = 0, 1$$

$$Y_k(z) = V_k(z^2) = \frac{1}{2}\left(X_k(z) + X_k(-z)\right)$$

$$= \frac{1}{2}\left(H_k(z)X(z) + H_k(-z)X(-z)\right) \qquad k = 0, 1.$$

Output signal:

$$\hat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z)$$

$$= \frac{1}{2}\left(H_0(z)F_0(z) + H_1(z)F_1(z)\right)X(z)$$

$$+ \frac{1}{2}\left(H_0(-z)F_0(z) + H_1(-z)F_1(z)\right)X(-z)$$

$$\therefore \quad 2\hat{X}(z) = \begin{bmatrix} X(z) & X(-z) \end{bmatrix} \underbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}_{\tilde{H}(z)} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$

known as the Aliasing Component (AC) matrix

Condition for Alias Cancellation.

- $H_0(z)$ is not ideal lowpass filter
- $H_1(z)$ is not ideal highpass filter
- after decimation by 2, aliasing is introduced
- $v_0[n]$ and $v_1[n]$ contain aliasing
- aliasing can be cancelled by careful choice of synthesis filters

We require

$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0$$

$$\therefore \quad F_0(z) = H_1(-z)$$

$$F_1(z) = -H_0(-z)$$

Amplitude and Phase Distortions.

If we write

$$\hat{X}(z) = T(z)\,X(z)$$

and assume the alias cancelling conditions are satisfied then

$$T(z) = \frac{1}{2}\Big(H_0(z)F_0(z) + H_1(z)F_1(z)\Big)$$

$$= \frac{1}{2}\Big(H_0(z)H_1(-z) - H_1(z)H_0(-z)\Big)$$

For no amplitude distortion we require

$$|T(z)|_{z=e^{jw}} = d \quad \leftarrow \text{a constant}$$

i.e. $T(e^{jw})$ is allpass

For no phase distortion we require

$$\angle T(z)\big|_{z=e^{jw}} = a + bw$$

a, b constants

i.e. $T(e^{jw})$ is linear phase.

Perfect Reconstruction

- alias cancellation
  (or no aliasing generated)
- no amplitude distortion
- no phase distortion

$$\hat{X}(z) = c\, z^{-n_0} X(z)$$

constant

Polyphase Representation

- now apply the polyphase representation to the QMF filter bank.

$$H_0(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

$$H_1(z) = E_0(z^2) - z^{-1} E_1(z^2)$$

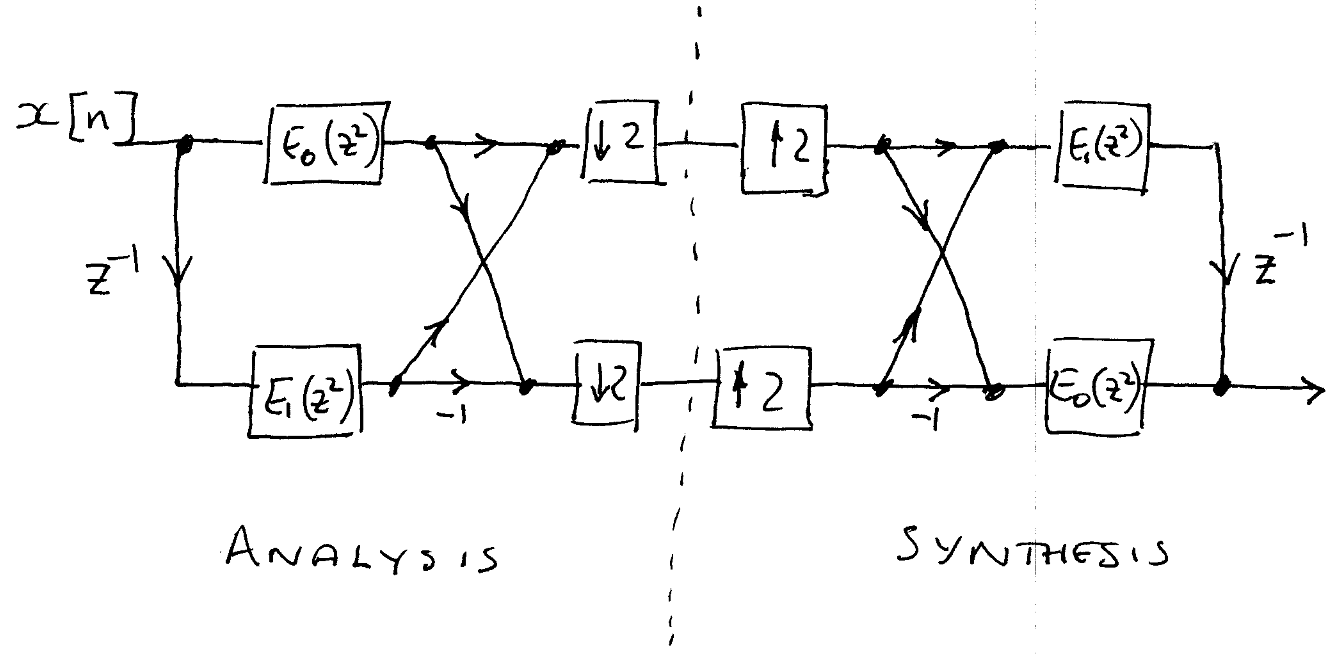
$$F_0(z) = E_0(z^2) + z^{-1} E_1(z^2)$$

$$F_1(z) = -E_0(z^2) + z^{-1} E_1(z^2)$$

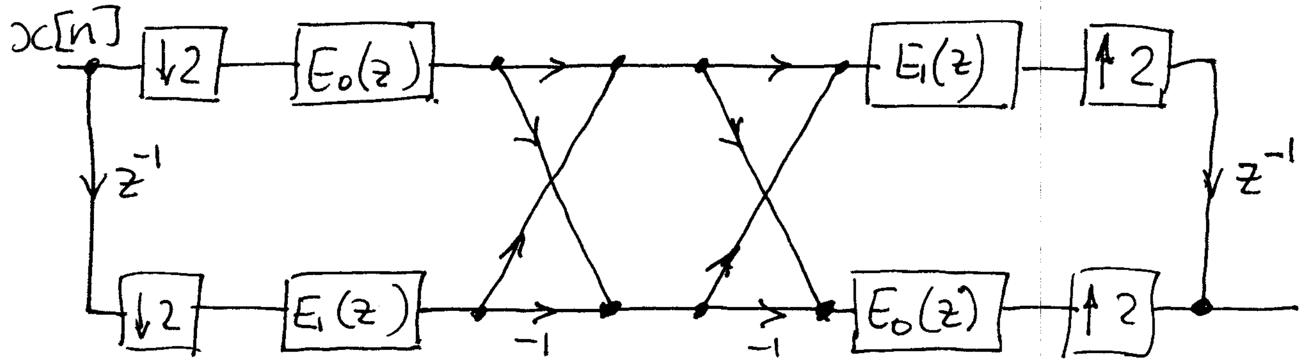
In matrix form:

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} E_0(z^2) \\ z^{-1} E_1(z^2) \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} z^{-1} E_1(z^2) \\ E_0(z^2) \end{bmatrix}$$



By application of the noble identities:



Note that all filtering is done at the lower sampling rate.