# Building Large Composition Tables via Axiomatic Theories

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## Abstract

The use of composition tables for efficiently representing and reasoning with jointly exhaustive pairwise disjoint sets of dyadic relations, is now well established in the AI literature. Whether typically built from axiomatic theories or from algebraic structures, most tables are built with a single theory in mind. We concentrate upon axiomatic theories for building these tables, and show how by factoring out related, but distinct formal theories (each capable of generating a composition table), large composition tables are easily constructed. This approach contrasts with the general difficulty of extracting out these tables where a parsimonious ontology and minimal number of primitives are used. We illustrate this with the construction of a non-trivial 20x20 composition table from two sub-theories supporting a 6x6 and 8x8 table. The ontological and representational ramifications for general theory building and the value of composition tables are discussed.

## 1 INTRODUCTION

The use of composition tables for efficiently representing and reasoning with Jointly Exhaustive and Pairwise Disjoint (*JEPD*) sets of n-ary relations, is now well established in AI literature. Composition tables reduce consistency checking of generated sets of ground instances of *JEPD* relations, to simple table look-up operations (Cohn, 1997). Whether typically built from axiomatic theories (e.g. Bennett, 1994; Randell, *et al*., 1992) or algebraic structures (e.g. Egenhofer, 1994), most tables are built with a single theory in mind. But extracting these tables can sometimes be difficult, particularly in the case of the former approach, where a parsimonious ontology and minimal number of primitives are used. In this paper we use axiomatic theories as our base, but (by developing a technique first described by Galton (Galton, 1994)) show how by factoring out related, but distinct formal theories (each capable of generating a composition table) large composition tables are easily constructed. We illustrate this approach with the construction of a non-trivial 20x20 composition table from two sub-theories respectively supporting a 6x6 and 8x8 table.

In section 2 we introduce composition tables in the light of axiomatic theories, from which they can be derived. In section 3, the general method of building larger tables is described, and the formal correctness of the method given. In section 4 we present an example target axiomatic theory to which the method is applied; factor out two embedded sub-theories, then show how the composition table is generated. Section 5 gives a program outline that directly implements the method to synthesise composition tables. Section 6 discusses *conceptual neighbourhoods*, defining the valid transitions between sets of *JEPD* relations. Section 7 presents an alternative axiomatisation for a generic set of occlusion relations. Section 8 describes a program suite, incorporating a resolution based theorem prover, which we used: (i) to cross check these results, and (ii) as a generic tool to assist in the task of theory-building itself. Finally we discuss the ontological and representational ramifications for general theory building and the value of composition tables.

## 2 COMPOSITION TABLES

An $n \times n$ composition table takes a set $n$ of mutually exhaustive and pair wise disjoint (*JEPD)* relations, and for each pair of relations $R_1(x,y)$ and $R_2(y,z)$, gives $R_3(x,z)$ as the set of all possible $R_3$ relations, implied by $R_1$ and $R_2$. In general $R_3$ will be a disjunction of entries; with the additional requirement that $R_3$ is the minimal set of instances implied by $R_1$ and $R_2$.

Composition tables are illustrated in Figures, 2, 3 and 6. Figure 6 gives the full composition table for the 20 *JEPD* relations of the Region Occlusion Calculus *ROC-20* (Randell, *et al*., 2001), which we discuss below. This calculus is used to model occlusion relations between arbitrary shaped bodies from a given viewpoint. Figures 2 and 3 show the composition tables for two related sub-theories *RCC-8* and *ROC-6*. If $R_1(A,B)$ and $R_2(B,C)$ hold, where $R_1$ is the specified relation along a row, and $R_2$ specified down a column, the respective cell entry at the intersection encodes the complete set of values for $R_3(A,C)$. In general, each entry for the table embodies a model, and in the case where the table has been generated from an axiomatic theory, a theorem of the underlying logic. While these tables can be extracted using model building and theorem proving techniques, the extraction is not necessarily straightforward. For example, difficulties working with the spatial theory *RCC*, led to a challenge for automated theorem provers, and motivated a tractable solution by respectively encoding sub-theories of RCC into intuitionistic logic (Bennett, 1994). This has led some to pursue alternative algebraic methods using the "n-intersection" method to factor out these compositions (Egenhofer, 1994; Egenhofer *et al*., 1994) and in a similar vein where composition tables are completely eliminated (El-Geresy and Abdelmoty, 1996).

Methods for generating composition tables currently fall into two main approaches: (i) the use of axiomatic theories, model-generation and theorem proving, and (ii) the use of algebraic structures and the intersection method. Either approach has its own particular merits. In the case of the axiomatic approach, the underlying *ontology* is highlighted and the reasoning about the domain can be applied not only to the entries of the table, but also to other *wff* that, as theorems, are not necessarily embodied in the table. For example, reasoning just using the composition table for *ROC-20* (to be described) is not sufficient to prove all the existential conditions entailed by the axiomatic theory, if completeness of the underlying theory is to be achieved. That is to say, given a model, we need to ask ourselves what set of truths are provable within our formal system. By contrast, the algebraic approach using the intersection method gains on the computational side. We argue that the method first discussed by Galton (1994) provides a practical alternative approach. While an axiomatic approach to model building is used, this is applied to sub-theories and their composition tables, where the larger theory and associated composition table is generated as a direct consequence.

Next we discuss the general method of large composition table construction, then we give an example using the theory *ROC-20* that embeds two sub-theories *RCC-8* and *ROC-6*.

# 3 GENERATING LARGE COMPOSITION TABLES

In his paper "Lines of Sight", Galton (1994) describes a method where by taking two independent formal theories, their composition tables, and a target set of relations constructible from these, a large composition table can be built. It is based on the assumption that each entry of the target composition table is defined in terms of the direct product of the two simpler sets of relations. We reformulate this result, by stating that given a target formal theory, one can build a composition table providing we have: (i) a set of *JEPD* definitions for the target theory, and for each sub-theory, (ii) a set of *JEPD* relations, and a composition table, and (iii) a set of *constraint axioms* that map predicates and functions defined in the target theory to predicates and functions in both sub-theories.

Let $\Psi$ and $\Sigma$ denote two sub-theories of the target theory $\Phi$, where $JEPD^{\Psi}$, $JEPD^{\Sigma}$ and $JEPD^{\Phi}$ are respectively their *JEPD* sets of relations, and where $Rn^{\Phi}(x,y) \in JEPD^{\Phi}$; similarly: $Rn^{\Psi}(x,y) \in JEPD^{\Psi}$ and $Rn^{\Sigma}(x,y) \in JEPD^{\Sigma}$. We also assume the existence of a function $\phi$, expressible in $\Phi$, that maps between $\Phi$ and $\Psi$. To create the larger composition table we wish to establish each $R_3$ relation set for (i), assuming (ii) to (iv) shown below, where schemas (i), (ii) and (iii) respectively encode the composition tables for theories $\Phi$, $\Psi$, and $\Sigma$, and where (iv) holds for each defined $Rn^{\Phi}(x,y)$ relation.

$$(i)\ R_1^{\Phi}(x,y)\ \&\ R_2^{\Phi}(y,z) \vdash R_3^{\Phi}(x,z)_1 \vee \ldots \vee R_3^{\Phi}(x,z)_n$$
$$(ii)\ R_1^{\Psi}(x,y)\ \&\ R_2^{\Psi}(y,z) \vdash R_3^{\Psi}(x,z)_1 \vee \ldots \vee R_3^{\Psi}(x,z)_n$$
$$(iii)\ R_1^{\Sigma}(x,y)\ \&\ R_2^{\Sigma}(y,z) \vdash R_3^{\Sigma}(x,z)_1 \vee \ldots \vee R_3^{\Sigma}(x,z)_n$$
$$(iv)\ Rn^{\Phi}(x,y) \leftrightarrow [Rn^{\Psi}(x,y)\ \&\ Rn^{\Sigma}(\phi(x),\phi(y))]$$

The method for generating the full, larger, composition table for $\Phi$ is as follows:

***Step1***: Using (iv), each single $Rn^{\Phi}$ entry is split into its two constituent relations: $Rn^{\Psi}$ and $Rn^{\Sigma}$.

***Step2**:* Using the composition tables encoded in (ii) and (iii); for each ordered pair $<R_1^n,R_2^n>$ of $Rn^{\Psi}$ and $Rn^{\Sigma}$ relations, we generate two $R_3^{\Psi}$ and $R_3^{\Sigma}$ sets, that together comprise all the possible relations (as decompositions) that can be formed.

***Step 3***: The third step simply re-builds the set of $R_3^{\Phi}$ entries from the generated set of $R_3^{\Psi}$ and $R_3^{\Sigma}$ relations. However, typically, not all combinations formed by taking one element from each $R_3^{\Psi}$ *and* $R_3^{\Sigma}$ set will have a model. This is guaranteed by ensuring that each $R_3^{\Psi}$ and $R_3^{\Sigma}$ combination satisfies the mapping axioms (incorporating $\phi$) of the theory. This leaves us with a maximal generated set of named $Rn^{\Phi}$ relations that populate the corresponding $R_3$ cell in the composition table.

The correctness of the method is easily shown:

**Step 1**: The translation from $Rn^{\Phi}(x,y)$ to its $Rn^{\Psi}(x,y)$ and $Rn^{\Sigma}(\phi(x),\phi(y))$ constituents, is true by definition, i.e. $Rn^{\Phi}(x,y) \equiv def.\ Rn^{\Psi}(x,y)\ \&\ Rn^{\Sigma}(\phi(x),\phi(y))$.

**Step 2**: The two composition tables for $\Psi$ and $\Sigma$, encoding the *theorems* of $\Phi$ guarantee that that each $R_3$ set of disjuncts, is both a logical consequence of $R_1$ and $R_2$, and exhaustively enumerates all the possible generated cases.

**Step 3**: The soundness of selecting only the $Rn^{\Psi}(x,y)$ and $Rn^{\Sigma}(\phi(x),\phi(y))$ relation pairs if $Rn^{\Phi}(x,y)$, follows immediately from the definitions and mapping axioms of the theory.

The application of steps 1-3 show that for each $R_1^{\Phi}$ and $R_2^{\Phi}$ pair, the result set $R_3^{\Phi}$ is both sound (i.e. is implied by the theory) and (by exhaustively generating all possible new $R_3^{\Phi}$ relations) is complete. **QED**.

We now develop our first-order theory to illustrate the general method.

# 4 ROC-20 AND RCC-8 - THE EXEMPLAR THEORIES

Our universe of discourse includes bodies, regions and points, all forming pairwise disjoint sets. A set of sorts and a *sorted logic* allowing *ad hoc* polymorphic functions and predicates to be handled, is assumed.

The notation and conventions used is as follows: ***type*** $a(\tau 1,.., \tau n): \tau n+1$ means function symbol $a$ is well sorted when its argument sorts are $\tau 1,.., \tau n$ with $\tau n+1$ as the result sort, and ***type*** $a(\tau 1,.., \tau n)$ means predicate $a$ is well sorted when defined on argument sorts $\tau 1,.., \tau n$. Axioms, definitions and theorems are respectively indicated as follows: *(A1,..,An), (D1,..,Dn), and (T1,..,Tn).* Where axiom/definitional schemas are used, the numbering in the parentheses reflects the number of object-level axioms and definitions generated.

We embed the mereo-topological theory *RCC-8* (Randell, *et al.*, 1992) into our theory, *ROC-20*. The same primitive dyadic relation *C/2*: '*C(x,y)*' read as "*x* is connected with *y*" is used. All the relations defined in *RCC-8* are used, and all carry their usual readings: *DC/2* (disconnected), *P/2* (part), *EQ/2* (equal), *O/2* (overlaps), *DR/2* (discrete) *PO/2* (partial overlap), *EC/2* (external connection), *PP/2* (proper part), *TPP/2* (tangential proper part), *NTPP/2* (non-tangential proper part). *PI/2, PPI/2, TPPI/2* and *NTPPI/2* are the inverse relations for *P/2*, *PP/2*, *TPP/2* and *NTPP/2*, respectively. Eight of these relations are provably *JEPD*, and are hereinafter referred to as $JEPD_{RCC\text{-}8}$.

Axioms for *C/2* and definitions for the dyadic relations of *RCC-8* are as follows:

*(A1)* $\forall x\ C(x,x)$
*(A2)* $\forall x \forall y\ [C(x,y) \rightarrow C(y,x)]$
*(A3)* $\forall x \forall y [\forall z[C(z,x) \leftrightarrow C(z,y)] \rightarrow EQ(x,y)]$[1]

*(D1)* $DC(x,y) \equiv def.\ \neg C(x,y)$
*(D2)* $P(x,y) \equiv def.\ \forall z[C(z,x) \rightarrow C(z,y)]$
*(D3)* $EQ(x,y) \equiv def.\ P(x,y)\ \&\ P(y,x)$
*(D4)* $O(x,y) \equiv def.\ \exists z[P(z,x)\ \&\ P(z,y)]$
*(D5)* $DR(x,y) \equiv def.\ \neg O(x,y)$
*(D6)* $PO(x,y) \equiv def.\ O(x,y)\ \&\ \neg P(x,y)\ \&\ \neg P(y,x)$
*(D7)* $EC(x,y) \equiv def.\ C(x,y)\ \&\ \neg O(x,y)$
*(D8)* $PP(x,y) \equiv def.\ P(x,y)\ \&\ \neg P(y,x)$
*(D9)* $TPP(x,y) \equiv def.\ PP(x,y)\ \&$
$\exists z[EC(z,x)\ \&\ EC(z,y)]$
*(D10)* $NTPP(x,y) \equiv def.\ PP(x,y)\ \&$
$\neg \exists z[EC(z,x)\ \&\ EC(z,y)]$
*etc.*

***type*** *Φ(Region,Region); where Φ ∈ {C,DC,P,EQ,O,DR,PO,EC,PP,TPP,NTPP,PI,PPI, TPPI,NTPPI}*

Assumed but not given here, is an axiom in *RCC-8* that guarantees every region has a non-tangential proper part *(A3)*, and a set of axioms *(A4-A9)* introducing Boolean functions for the sum, complement, product, difference of regions, the universal spatial region; and an axiom that introduces the sort *Null* enabling partial functions to be handled – see (Randell, *et al.*, 1992) for more details.

## 4.1 MAPPING FUNCTIONS AND AXIOMS

*ROC-20* uses *RCC-8* to model the spatial relationship between bodies, volumes, and their corresponding images with respect to a viewpoint. The formal distinction is maintained by introducing two functions: '*region(x)*' read as "the region occupied by *x*" and '*image(x,v)*' read as "the image of *x* with respect to viewpoint *v*". The function: *region/1*, maps a body to the volume of space it occupies, and *image/2* maps a body and a viewpoint to its image; i.e. the region defined by the set of projected half-lines originating at the viewpoint and intersecting the body, so forming part of the surface of a sphere of infinite radius centred on the viewpoint. A set of axioms acting as

[1]Strictly speaking axiom (*A3)* is immediate consequence of definitions *(D2)* and *(D3)*, but is added here simply to clarify the relationship between the relations *C/2* and *EQ/2*.

a set of *spatial constraints* between bodies, a given viewpoint, and their corresponding images are given[2]:

$$(A11\text{-}A15)\ \forall x \forall y\ [\Phi(region(x),region(y)) \rightarrow \forall v\ [\Phi(image(x,v),image(y,v))]]$$

***type*** *region(Body):Region*[3]
***type*** *image(Body,Point):Region*
***type*** *Φ(Region,Region) where:*
$\Phi \in \{C,O,P,NTPP,EQ\}$[4]

### 4.2 OCCLUSION

For the occlusion part of the theory, a second primitive relation: '*TotallyOccludes(x,y,v)*', read as "*x* totally occludes *y* with respect to viewpoint *v*", is introduced. *Totally Occludes/3* is axiomatised to be transitive. Several other axioms are used to embed *RCC-8* into this theory, making *TotallyOccludes/3* additionally asymmetrical and irreflexive.

The intended *geometric* meaning of total occlusion is as follows. Let *line(p1,p2,p3)* mean that points *p1, p2* and *p3* fall on a straight line with *p2* strictly between *p1* and *p3*. Then, *x* totally occludes *y* from *v iff* for every point *p* in y, there exists a point *q* in *x* such that *line(v,q,p)*, and there are no points *p′* in *y*, and *q′* in *x*, such that *line(v,p′,q′)*. An object *x* can totally occlude an object *y* even if *x* itself is totally occluded by another object.

Axiom *A16* (below) states that if *x* totally occludes *y*, *x* totally occludes any part of *y*; and *A17* if *x* totally occludes *y* no part of *y* totally occludes part of *x*. *A17* excludes cases of total occlusion where part of the occluding wraps 'behind' the occluded object. This is an example of *mutual occlusion*, and which is defined below in definition (*D17*). *A18* states that if *x* totally occludes *y*, the image of *x* subtends the image of *y*. Note that *A18* is not a biconditional because the *P/2* relation (defined on images here) is indifferent to various factors including relative distance and overlap between occluding bodies in the assumed model. Spatial identity of regions in terms of co-location still applies, but is restricted to the dimensionality of the regions being modelled.

$$(A16)\ \forall x \forall y \forall z \forall v\ [[TotallyOccludes(x,y,v)\ \&\ TotallyOccludes(y,z,v)] \rightarrow TotallyOccludes(x,z,v)]$$

$$(A17)\ \forall x \forall y \forall v\ [TotallyOccludes(x,y,v) \rightarrow \forall zu[[P(region(z),region(x))\ \&\ P(region(u),region(y))] \rightarrow \neg TotallyOccludes(u,z,v)]]$$

$$(A18)\ \forall x \forall y \forall v\ [TotallyOccludes(x,y,v) \rightarrow P(image(y,v),image(x,v))]$$

***type*** *TotallyOccludes(Body,Body,Point)*

Total occlusion between distinct bodies implies occlusion (*T1*), which in turn implies region overlap between their corresponding images (*T2*). Moreover, we can also show that if *x* totally occludes *y*, *x* totally occludes every part of *y* (*T3*) and that from every viewpoint, body *x* has two parts (*y* and *z)* such that the one part (*y)* totally occludes the other part (*z*). This can be interpreted in a 3D model to mean that bodies have *depth* (*T4*):

$$(T1)\ \forall x \forall y \forall v[TotallyOccludes(x,y,v) \rightarrow Occludes(x,y,v)]$$

$$(T2)\ \forall x \forall y \forall v\ [Occludes(x,y,v) \rightarrow O(image(x,v),image(y,v))]$$

$$(T3)\ \forall x \forall y \forall v[TotallyOccludes(x,y,v) \rightarrow \forall z[P(region(z),region(x))] \rightarrow TotallyOccludes(x,z,v)]$$

$$(T4)\ \forall x \forall v \exists y \exists z[P(region(y),region(x))\ \&\ P(region(z),region(x))\ \&\ TotallyOccludes(x,y,v)]$$

A refined set of occlusion relations is defined including weak occlusion, and partial and mutual occlusion. '*Occludes(x,y,v)*' is read as "*x* occludes *y* from viewpoint *v*" and means from *v* that some part of *x* totally occludes some part of *y*. *Occludes/3* in contrast to *O/2* is *non-symmetrical*. Other more specific occlusion relations are defined and then mapped to their *RCC-8* analogues. For completeness (not listed here) inverse relations are given for *Occludes/3*, *TotallyOccludes/3* and *PartiallyOccludes/3* (*D18-D20)*; leaving the null case: *NonOccludes/3*, where no occlusion arises. The six relations: *NonOccludes/3*, *MutuallyOccludes/3*; and *TotallyOccludes/3*, *PartiallyOccludes/3*, and their inverses also form another *JEPD* set - $JEPD^{ROC\text{-}6}$. For more details see (Randell *et al*., 2001).

$$(D15)\ Occludes(x,y,v) \equiv_{def.} \exists z \exists u[P(region(z),region(x))\ \&\ P(region(u),region(y))\ \&\ TotallyOccludes(z,u,v)]$$

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[2] Although not developed here, the distinction made between bodies and regions enables one to define the notion of free space and model spatial occupancy – see (Shanahan, 1996).

[3] Sortal declarations given here are not as restricted as they could be, for example we could declare: ***type*** *region(Body): 3DRegion*, and ***type*** *image(Body,Point): 2DRegion***,** where *2DRegion* and *3DRegion* are (disjoint) subsorts of the sort *Region.*

[4] The set of predicate constants used here (and in the set of axioms for *ROC-20* that appear in this paper) differs from that presented in (Randell *et al*., 2001) where redundancy in the original set of axioms has been addressed. The exception is the removal of axiom (*A13)* in that paper, for which counter-examples have been found. We wish to thank Antony Galton for bringing our attention to this.

(D16) $PartiallyOccludes(x,y,v) \equiv def.$
$Occludes(x,y,v)\ \&$
$\neg TotallyOccludes(x,y,v)\ \&$
$\neg Occludes(y,x,v)$
(D17) $MutuallyOccludes(x,y,v) \equiv def.$
$Occludes(x,y,v)\ \&\ Occludes(y,x,v)$
(D21) $NonOccludes(x,y,v) \equiv def.$
$\neg Occludes(x,y,v)\ \&\ \neg Occludes(y,x,v)$

(A19) $\forall x \forall y \forall v[NonOccludes(x,y,v) \rightarrow DR(image(x,v),image(y,v))]$
(A20) $\forall x \forall y \forall v[PartiallyOccludes(x,y,v) \rightarrow [PO(image(x,v),image(y,v)) \vee PP(image(x,v),image(y,v))]]$
(A21) $\forall x \forall y \forall v[MutuallyOccludes(x,y,v) \rightarrow O(image(x,v),image(y,v))$

**type** $\Phi(Body,Body,Point)$; where $\Phi \in \{Occludes, PartiallyOccludes, MutuallyOccludes, NonOccludes, ...\}$

Finally, a total set of 20 *JEPD* relations, $JEPD^{ROC\text{-}20}$ is defined using more specific instances on the *P/2* relation. These are generated using the following definitional schemas, and are illustrated with a graphical model shown in Figure 1. This also provides a key to the 5x4 matrices that populate the cell entries for the 20x20 composition table illustrated in Figure 6. In each case a filled/unfilled square, respectively indicates a model/no model. This then completes the development of the basic theory that is sufficient for our purposes here.

(D22-D33) $\Phi\Psi(x,y,v) \equiv def.$
$\Phi(x,y,v)\ \&\ \Psi(image(x,v),image(y,v))$
(D34-D41) $X\Psi^{-1}(x,y,v) \equiv def.$
$X(y,x,v)\ \&\ \Psi(image(y,v),image(x,v))$

***type*** $\Phi\Psi\ (Body,Body,Point)$
***type*** $X\Psi^{-1}(Body,Body,Point)$
***type*** $\Phi(Body,Body,Point)$
***type*** $X\ (Body,Body,Point)$
***type*** $\Psi\ (Region,Region)$

where if:

$\Phi$ = *NonOccludes, then* $\Psi \in \{DC,EC\}$
$\Phi$ = *PartiallyOccludes, then* $\Psi \in \{PO,TPP,NTPP\}$
$\Phi$ = *TotallyOccludes, then* $\Psi \in \{EQ,TPPI,NTPPI\}$
$\Phi$ = *MutuallyOccludes, then* $\Psi \in \{PO,EQ,TPP,NTPP\}$

and where if:

$X$ = *PartiallyOccludes, then* $\Psi \in \{PO,TPP,NTPP\}$
$X$ = *TotallyOccludes, then* $\Psi \in \{EQ,TPPI,NTPPI\}$
$X$ = *MutuallyOccludes, then* $\Psi \in \{TPP,NTPP\}$

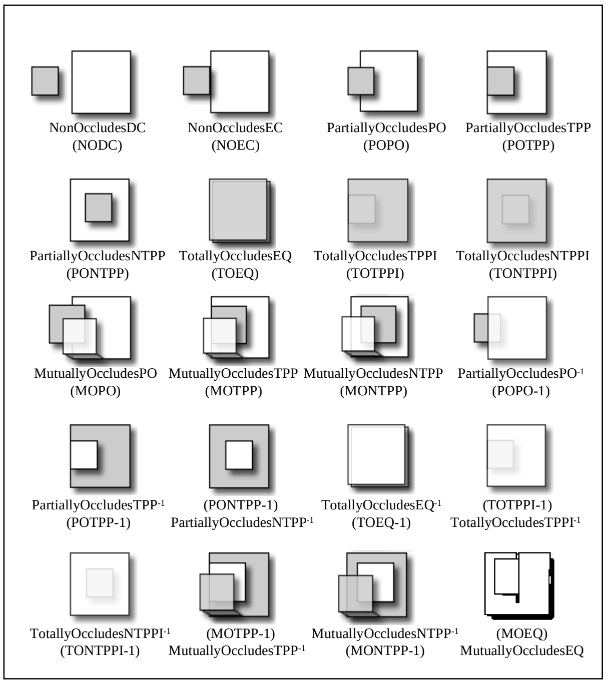


**Figure 1**: Graphical model for the *ROC-20* relations.

## 5 GENERATING THE 20X20 COMPOSITION TABLE

The method used to compute the $20 \times 20$ composition table exploits that outlined by Galton. In our case it is based on the fact that each *ROC-20* relation can be defined in terms of the direct product of two simpler (i.e. *ROC-6* and *RCC-8*) sets of relations; the composition table for the product set being directly generated from the composition tables of its factors. Where *RCC-8* considers modes of connection between regions, *ROC-6* considers generic occlusion relationships between bodies with respect to a viewpoint.

The composition table for *ROC-6* is shown in Figure 3, the relation names: NO, PO, TO, MO, POI and TOI, respectively abbreviate: *NonOccludes*, *PartiallyOccludes*, *TotallyOccludes*, *MutuallyOccludes*, and the two inverse relations: $PartiallyOccludes^{-1}$ and $TotallyOccludes^{-1}$.

|  | DC | EC | PO | TPP | TPPI | EQ | NTPP | NTPPI |
|---|---|---|---|---|---|---|---|---|
| DC | DC EC PO TPP TPPI EQ NTPP NTPPI | DC EC PO TPP NTPP | DC EC PO TPP NTPP | DC EC PO TPP NTPP | DC | DC | DC EC PO TPP NTPP | DC |
| EC | DC EC PO TPPI NTPPI | DC EC PO TPP TPPI EQ | DC EC PO TPP NTPP | EC PO TPP NTPP | DC EC | EC | PO TPP NTPP | DC |
| PO | DC EC PO TPPI NTPPI | DC EC PO TPPI NTPPI | DC EC PO TPP TPPI EQ NTPP NTPPI | PO TPP NTPP | DC EC PO TPPI NTPPI | PO | PO TPP NTPP | DC EC PO TPPI NTPPI |
| TPP | DC | DC EC | DC EC PO TPP NTPP | TPP NTPP | DC EC PO TPP TPPI EQ | TPP | NTPP | DC EC PO TPPI NTPPI |
| TPPI | DC EC PO TPPI NTPPI | EC PO TPPI NTPPI | PO TPPI NTPPI | PO TPP TPPI EQ | TPPI NTPPI | TPPI | PO TPP NTPP | NTPPI |
| EQ | DC | EC | PO | TPP | TPPI | EQ | NTPP | NTPPI |
| NTPP | DC | DC | DC EC PO TPP NTPP | NTPP | DC EC PO TPP NTPP | NTPP | NTPP | DC EC PO TPP TPPI EQ NTPP NTPPI |
| NTPPI | DC EC PO TPPI NTPPI | PO TPPI NTPPI | PO TPPI NTPPI | PO TPPI NTPPI | NTPPI | NTPPI | PO TPP TPPI EQ NTPP NTPPI | NTPPI |

**Figure 2**: Composition table for *RCC-8*

|  | NO | PO | POI | TO | MO | TOI |
|---|---|---|---|---|---|---|
| NO | NO PO POI TO MO TOI | NO PO POI MO TOI | NO PO POI MO TOI | NO | NO PO POI MO TOI | NO PO POI MO TOI |
| PO | NO PO POI TO MO | NO PO POI TO MO | NO PO POI TO MO TOI | NO PO TO | NO PO POI TO MO TOI | PO POI MO TOI |
| POI | NO PO POI TO MO | NO PO POI TO MO TOI | NO PO POI MO TOI | NO PO POI TO MO | NO PO POI MO TOI | POI MO TOI |
| TO | NO PO POI TO MO | PO TO MO | PO POI TO MO | TO | PO TO MO | PO POI TO MO TOI |
| MO | NO PO POI TO MO | NO PO POI TO MO | NO PO POI TO MO TOI | NO PO POI TO MO | NO PO POI TO MO TOI | POI MO TOI |
| TOI | NO | NO PO POI MO TOI | NO POI TOI | NO PO POI TO MO TOI | NO PO POI MO TOI | TOI |

**Figure 3**: Composition table for *ROC-6*

We established the relative consistency and correctness of the composition table entries for *ROC-6* by interpreting the relations against a graphical model that satisfied the axioms of the theory. That is to say a handcrafted model was generated for each of the (156) individual $R_1^6$, $R_2^6$ and $R_3^6$ cell entries. This result was then independently checked against the output from a batch program that used the resolution theorem prover *SPASS* - requiring a total of $6^3 = 216$ potential theorems to check (Section 8). In the case of *RCC-8*, the composition table was taken as a given – see Bennett (1994).

## 5.1 A WORKED EXAMPLE

Let: $Rn^{20}(x,y,v) \in JEPD^{ROC\text{-}20}$ be an instance of $Rn^{\Phi}(x,y) \in JEPD^{\Phi}$; similarly let: $Rn^6(x,y,v) \in JEPD^{ROC\text{-}6}$ be an instance of $Rn^{\Psi}(x,y) \in JEPD^{\Psi}$ and $Rn^8(x,y) \in JEPD^{RCC\text{-}8}$ of $Rn^{\Sigma}(x,y) \in JEPD^{\Sigma}$. The function *image/2* substitutes for $\phi$. Therefore, we wish to establish each $R_3^{20}$ relation set for (i), assuming (ii) to (iv):

$$(i)\ R_1^{20}(x,y,v)\ \&\ R_2^{20}(y,z,v) \vdash R_3^{20}(x,z,v)_1 \vee \ldots \vee R_3^{20}(x,z,v)_n$$

$$(ii)\ R_1^{6}(x,y,v)\ \&\ R_2^{6}(y,z,v) \vdash R_3^{6}(x,y,v)_1 \vee \ldots \vee R_3^{6}(x,z,v)_n$$

$$(iii)\ R_1^{8}(x,y)\ \&\ R_2^{8}(y,z) \vdash R_3^{8}(x,z)_1 \vee \ldots \vee R_3^{8}(x,z)_n$$

$$(iv)\ Rn^{20}(x,y,v) \leftrightarrow [Rn^{6}(x,y,v)\ \&\ Rn^{8}(image(x,v),image(y,v))]$$

where, as before, the schemas (i), (ii) and (iii) respectively encode the composition tables for *ROC-20* ($\Phi$), *ROC-6* ($\Psi$), and *RCC-8* ($\Sigma$), where (iv) holds for each defined $Rn^{20}(x,y,v)$ relation. In the case of *ROC-20* and *ROC-6* each *JEPD* relation is ternary, but where the last argument acts only as an index term.

In the following example, a cell entry for *ROC-20's* composition table is derived and justified against the theory. The three steps highlighted mirror those given in section 3. We wish to compute the cell entries for *R3*, where:

*R1* = $R_1^{20}$ = *TotallyOccludesTPPI(a,b,v1)*
*R2* = $R_2^{20}$ = *PartiallyOccludesPO(b,c,v1)*

**Step 1:** Unpack the clauses using the definitions.

*R1 = {TotallyOccludes(a,b,v1),*
*TPPI(image(a,b,v1),image(b,c,v1))}*
*R2 = {PartiallyOccludes(b,c,v1),*
*PO(image(b,c,v1),image(b,c,v1))}*

**Step 2:** Compute the composition table entries (atoms here) for each paired set of relations:

*<TotallyOccludes(a,b,v), PartiallyOccludes(b,c,v1)> =*
*{PartiallyOccludes(a,c,v1),*
*TotallyOccludes(a,c,v1),*
*MutuallyOccludes(a,c,v1)}*

*<TPPI(image(a,b,v1),image(b,c,v1)),*
*PO(image(b,c,v1),image(b,c,v1))> =*
*{PO(image(a,v1),image(c,v1)),*
*TPPI(image(a,v1),image(c,v1)),*
*NTPPI(image(a,v1),image(c,v1))}*

**Step 3:** Take the cross-product between both sets of generated atoms, and rebuild those satisfying the definitions and any other constraints arising from the underlying theory:

*(A22)* $\forall x \forall y \forall v\ [TotallyOccludes(x,y,v) \rightarrow P(image(y,v),image(x,v))]$

rules out:

*{TotallyOccludes(a,c,v1),PO(image(a,v1),image(c,v1))},*

owing to the theorem:

$$\forall x \forall y\ [PI(x,y) \leftrightarrow [EQ(x,y) \vee TPPI(x,y) \vee NTPP1(x,y)]],$$

(A24) $\forall x \forall y \forall v[PartiallyOccludes(x,y,v) \rightarrow [PO(image(x,v),image(y,v)) \vee PP(image(x,v),image(y,v))]]$

rules out:

*{PartiallyOccludes(a,c,v1),TPPI(image(a,v1),image(c,v1))},*
*{PartiallyOccludes(a,c,v1),NTPPI(image(a,v1),image(c,v1))},*

owing to the theorem:

$$\forall x \forall y\ [PP(x,y) \leftrightarrow [TPP(x,y) \vee NTPP(x,y)]]$$

(A25) $\forall x \forall y \forall v[MutuallyOccludes(x,y,v) \rightarrow [PO(image(x,v),image(y,v)) \vee P(image(x,v),image(y,v)) \vee PI(image(x,v),image(y,v))]]$

no cases ruled out.

*Hence: R3 = {PartiallyOccludesPO(a,c,v,1), TotallyOccludesTPPI(a,c,v,1), TotallyOccludesNTPPI(a,c,v,1), MutuallyOccludesPO(a,c,v1),* $MutuallyOccludesTPP^{-1}(a,c,v1)$, $MutuallyOccludesNTPP^{-1}(a,c,v1)$*}*

### 5.2 THE PROGRAM

The full *ROC-20* composition table (including the graphical output) shown in Figure 6 was automatically generated using a program that implements the method described in section 3. Each 5x4 matrix (as the intersection of a row and column) encodes a $R_3^{20}$ disjunctive entry, for a given $R_1^{20}$ and $R_2^{20}$ pair. A filled/unfilled square respectively represents a single $R_3^{20}$ disjunction that has a model/no model. The 20x20 composition table results were also confirmed using the resolution theorem prover *SPASS* [http://spass.mpi-sb.mpg.de/] – which is discussed further below.

Let `tab1`, `tab2` and `tab3` each be three-dimensional arrays encoding, respectively, the two smaller composition tables, Ψ (*ROC-6*) and Σ (*RCC-8*), and the larger composition table Φ (*ROC-20*) to be synthesised. The first and second indices select a cell by row and column, the third an element in a cell. The size of each table will be `size1`, `size2` and `size3`.

Let `deftab` be the table of $Rn^{\Phi}$ entries defining the constituent pairings between the two smaller theories ($Rn^{\Psi}$ and $Rn^{\Sigma}$). By definition the dimensions of `deftab` are [`size3`,`2`]. From section 4.3 `deftab` is initialised thus:

```
deftab := {{NO,DC}, {NO,EC}, {PO,PO}, {PO,TPP},
  {PO,NTPP}, {TO,EQ}, {TO,TPPI}, {TO,NTPPI},
  {MO,RPO}, {MO,TPP}, {MO,NTPP}, {POI,PO},
  {POI,TPPI}, {POI,NTPPI}, {TOI,EQ}, {TOI,TPP},
  {TOI,NTPP}, {MO,TPPI}, {MO,NTPPI}, {MO,EQ}};
```

This corresponds to step 1 of section 3. (Note the use of program constants `POI` and `TOI` to denote the inverse relations defined in *D34-41*.)

The psuedo-code for the table synthesis procedure is given below:

```
BuildTableBySynthesis(deftab, tab1, tab2, tab3,
                              size1, size2, size3)
  {
  clear(tab3);
  for (i := 1 to size3)
    {
    for (j := 1 to size3)
      {
      cell1[] := tab1[deftab[i,1],deftab[j,1]];
      cell2[] := tab2[deftab[i,2],deftab[j,2]];
      for (k := 1 to size1)
        {
        for(l := 1 to size2)
          {
          for (m := 1 to size3)
            {
            if (cell1[k] = deftab[m,1] AND
                cell2[l] = deftab[m,2])
              {
              tab3[i,j,m] := 1;
              }
            }
          }
        }
      }
    }
  }
```

The loops (**`i`**, **`j`**, **`k`** and **`l`**) sequentially generate every possible ordered pair <$R_1^n$,$R_2^n$> (step 2 of section 3). The innermost loop (**`m`**) checks whether each combination so generated satisfies the mapping axioms (as encoded in `deftab`) and populates the equivalent entry in the large composition table `tab3` if it is satisfied (step 3).

## 6 CONCEPTUAL NEIGHBOURHOODS

An inspection of the composition tables for *ROC-6* and *ROC-20*, shows that each $R_3$ entry forms a *conceptual neighbourhood* (Freksa, 1992), i.e. each set of elements forms a connected subset of relations for each corresponding neighbourhood diagram.

A neighbourhood diagram for *ROC-20* reworked as an *envisionment table* is given in (Randell *et al.*, 2001), but this is easily generated from the map between the two neighbourhood diagrams for $JEPD^{RCC\text{-}8}$ and $JEPD^{ROC\text{-}6}$ as follows. Let: $Rn^{20}(x,y,v) \in JEPD^{ROC\text{-}20}$; $Rn^{6}(x,y,v) \in$

$JEPD^{ROC\text{-}6}$ and $Rn^8(x,y) \in JEPD^{RCC\text{-}8}$; where for each $Rn^{20}$ definition: $Rn^{20}(x,y,v) \leftrightarrow [Rn^6(x,y,v)\ \&\ Rn^8(image(x,v),image(y,v))]$. Then given the pair of relations: $(R_1^{20}(x,y,v),\ R_2^{20}(x,y,v))$ a path exists in the neighbourhood diagram *for ROC-20 iff* a neighbourhood path exists between: $(R_1^8(x,y),\ R_2^8(x,y))$ *and* $(R_1^6(x,y,v),\ R_2^6(x,y,v))$.

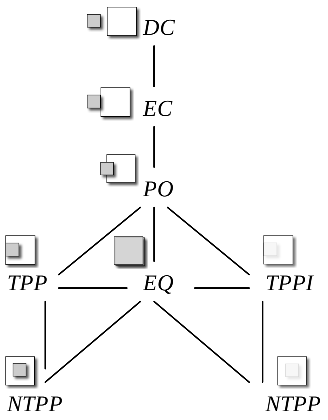


**Figure 4**: Neighbourhood diagram for RCC-8

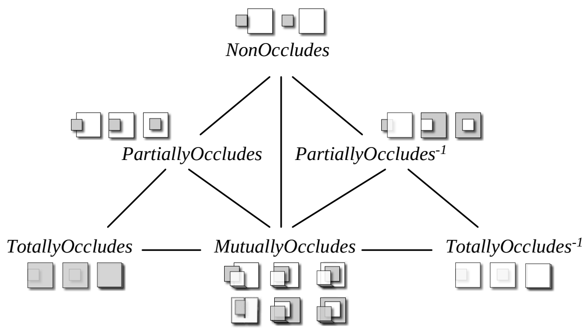


**Figure 5**: Neighbourhood diagram for ROC-6

## 7 ROC-6 AND RCC-5

Given the fact that *ROC-6* does not use any *RCC-8* relation greater than the part-whole relation *P/2*, a formal relationship between *ROC-6* and the weaker mereo-topological theory *RCC-5* (Bennett, 1994) is naturally suggested. However, even though we can axiomatise *RCC-5*, the formal relationship between the two is not a simple matter of subsumption.

To axiomatise *RCC-5*, the stronger primitive dyadic overlap relation *O/2* (c.f. *C/2* for *RCC-8*) is used. '*O(x,y)*' is read as "*x* overlaps *y*", meaning that a region is shared in common. Many of the relations named in *RCC-8* are used, and all carry the same readings and sortal declarations: *P/2* (part), *EQ/2* (equal), *DR/2* (discrete) *PO/2* (partial overlap), *PP/2* (proper part), with the inverse relations *PI/2, and PPI/2*. Of these, five (i.e. *PO/2*, *PP/2*,*EQ/2*, *PP-1/2*, and *DR/2*} are *JEPD*.

Axioms for *O/2*, definitions for the dyadic relations of *RCC-5*, and the mapping axioms are now restricted to the predicates {O,P,EQ}, which follow those for *ROC-20*. The exception is the P/2 relation, which is now defined directly in terms of *O/2* and not *C/2* as before:

(A1') $\forall x\, O(x,x)$
(A2') $\forall x \forall y\, [O(x,y) \rightarrow O(y,x)]$
(A3') $\forall x \forall y [\forall z[O(z,x) \leftrightarrow O(z,y)] \rightarrow EQ(x,y)]$

(D2') $P(x,y) \equiv_{def.} \forall z[O(z,x) \rightarrow O(z,y)]$
*etc.*

Again mirroring *RCC-8*, but not reproduced here, is an axiom in *RCC-5* that guarantees every region has a proper part, and a set of axioms replicating the same Boolean functions used in *RCC-8*. The only difference between the axioms used in *RCC-8* compared with their analogues in *RCC-5*, is that '*C(x,y)*' appearing in clauses in the axiomatisation for *RCC-8* is substituted with '*O(x,y)*'; with the single exception of the complement function *compl(x)*. This changes from: $\forall x \forall y[[C(x,compl(y)) \leftrightarrow \neg NTPP(x,y)]\ \&\ [O(x,compl(y)) \leftrightarrow \neg P(x,y)]]$, to: $\forall x \forall y\, [O(x,compl(y)) \leftrightarrow \neg P(x,y)]$. This change has an important formal consequence that is discussed below. From here on, we simply mirror axioms (A16-21), definitions (D15-21), and the mapping axioms described in section 4.1, but restricted to the set {*O,P,EQ*} governing the $JEPD^{\ ROC\text{-}6}$ set of relations, yielding the theory *ROC-6*.

One cannot simply reduce *RCC-8*, as presented in (Randell, *et al*., 1992), to *RCC-5* by the simple addition of the 'reduction' axiom: $\forall x \forall y[C(x,y) \rightarrow O(x,y)]$, (i.e. meaning external connection between regions is not allowed in the domain) without contradiction. The restriction stems from the complement axiom:

$$\forall x \forall y[[C(x,compl(y)) \leftrightarrow \neg NTPP(x,y)]\ \& \\ [O(x,compl(y)) \leftrightarrow \neg P(x,y)]].$$

With *C/2* stipulated to be equivalent to *O/2*, *TPP/2* can never be true in the domain, and *PP/2* becomes equivalent to *NTPP/2*; meaning $\forall x[O(x,compl(x)) \leftrightarrow \neg PP(x,x)]$ follows. But given $\forall x \neg PP(x,x)$ is true, each region overlaps (i.e. shares a region in common with) its complement – which leads to the contradiction.

## 8 THEORY CORROBORATION

In addition to computing the 20x20 entries using the program described in section 5.2 we also confirmed the results against output provided by the resolution theorem proving program *SPASS*. In both cases each of the *JEPD* relations were encoded as bitmaps; while in the case of the latter, a customised shell program was developed that interfaced to the theorem prover functioning as a flexible, general purpose *theory development tool*.

First, all the axioms and definitions of *ROC-20* were coded up in SPASS notation. Then the program steps were as follows. (i) We incrementally checked the axiom set for possible redundancy, by trying to prove each axiom in turn as a theorem. Axioms in the set that proved to be redundant were simply flagged. (ii) We automatically generated a set of clauses (in SPASS notation) encoding the set of $JEPD^{ROC\text{-}6}$ relations and tested that they were indeed, jointly exhaustive and pairwise disjoint. These clauses were then appended to the original axiom set, solely for the purposes of program efficiency. (iii) A set of ($6^3$ = 216) clauses which encoded potential theorems for each composition table entry were similarly machine generated and batch processed, and on completion were again added to the SPASS axiom set. This completed the run for *ROC-6*. For *ROC-20* the same steps were repeated: (iv) first testing that the set of $JEPD^{ROC\text{-}20}$ relations were *JEPD*, and finally (v) automatically constructing each composition table entry for *ROC-20* requiring $20^3$ (8,000) potential theorems to be checked. In order to reduce the overall run time, the time-out allocated for the individual clauses to be proved was weighted in favour of predicted theorems. This was determined by the results generated by the 'synthesised composition table generation' method.

Despite the obvious limitation imposed by the implemented logic, i.e. using a *semi-decidable system* (meaning no practical distinction can be made between distinguishing between non-theorems, or theorems not yet proved) the program nevertheless threw up several interesting surprises. The first was revealing redundancy in several *ROC-20* axioms that had hitherto escaped our attention. This was despite an increasing familiarity we were building up with the developing theory. This pruning, in turn, helped us to identify a simpler abstract graphical model for the theory. Secondly, after only a short period into a few program runs, we soon noticed anomalous results that could not be accounted for in our model. On closer inspection the anomaly was quickly traced to an incorrect data entry error in the composition table for *RCC-8.* That this appeared very quickly, highlighted two things about this approach: (i) by virtue of the modular nature of the shell program, the correction was easily done, leaving only background processing time for a complete program re-run; and (ii) increasing confidence in the formal correctness of the developing theory was gained, as more clauses were incrementally added without showing any sign of a problem on the interpreted output. Finally, it is noteworthy that when using an earlier set of relations, a simple *JEPD* test failed to prove the relation set was exhaustive. This pointed to the existence of a 'missing' node (*MOEQ/2*) in the embedding relational lattice and that too strong a model had been used to interpret the theory; which indeed proved to be the case. In each case the flexibility of the shell program has proved to be a very useful tool for the process of theory development.

## 9 WORKING METHODOLOGY AND ONTOLOGICAL ISSUES.

This approach described here, gains over the naïve method of generating a large composition table from a single unified theory in several ways. Firstly, by factoring out sub-theories, it clarifies the underlying ontology, and how these sub-theories map between each other. Secondly, with the theoretical and practical need to compare theories, the modular approach allows existing theories to be re-used rather than building yet another axiomatic theory whose only connection with an existing theory may be implicit from the assumed model and informal semantics given.

There are two approaches we can take. Top down, we take a single theory, identify and factor out subsets of n-ary *JEPD* relations, embed these in relational lattices and extract out smaller compositions tables for each. In contrast, building bottom up, while we can generate the composition table for the larger theory from two (or more) sub-theories, the consequence classes for both will not be necessarily identical. While some reasoning tasks using composition tables are adequate, e.g. satisfiability checking of sets of atomic propositions, not all are. Clearly, composition tables (encoding a set of *universal axioms*) cannot capture all the *existential conditions* of the underlying theory; but neither do they encode all the universal relational properties of the defined predicates. For example, the set of universal axioms used to define the composition table for *RCC-8* and those encoding their *JEPD* properties, is not sufficient to prove the symmetry of the relations *C/2* and *O/2*.

## 10 CONCLUSIONS AND FUTURE WORK

An open question remains about how much of the above method can be automated. Currently, we hand-build our target set of defined *JEPD* relations, but it is certainly possible to machine generate these given a weaker *JEPD* superset of defined relations, and then for each pair of relations from that set, automatically checking subsumption and pairwise disjoint relationships. Currently we use a customised implementation of the resolution refutation program SPASS that takes an existing axiomatic theory, and automatically builds and checks lemmas (specifically *JEPD* and composition table information) and redundancy of the axioms used. This concentrates specifically upon the task of building, as opposed to implementing and *using* such theories in an applied domain, e.g. using *ROC-20* in a real-world Cognitive Robotics programme.

One promising approach arises simply from being able to factor out and map between identified sub-theories and their respective composition tables. This metalogical structure and decomposition naturally leads into parallel or hierarchical search techniques. For example, using *ROC-20*, we can map *wff* defined in *ROC-20* (modelling bodies) to *wff* defined in *RCC-8* (their corresponding images), restricting the refutation search to a hierarchically and increasingly constrained set of *RCC-8*, *ROC-6* or *ROC-20* formulae, or alternatively to conduct the search in parallel.

From a theoretical standpoint, more work is required to see exactly what the conceptual neighbourhood property gives with respect to the task of showing your axiomatic theory is both correct and complete, for the intended domain (c.f. Duentsch *et al*., 1998).

The method illustrated here, is very general and can be easily extended to build very large formal theories by mapping between two or more sub-theories, each of which gives rise to a set of *JEPD* relations and their associated composition table. Axioms that map between predicates and functions between the theories are singled out, and used as (but also serve to partition the theory into) sub-theories and the map between them.

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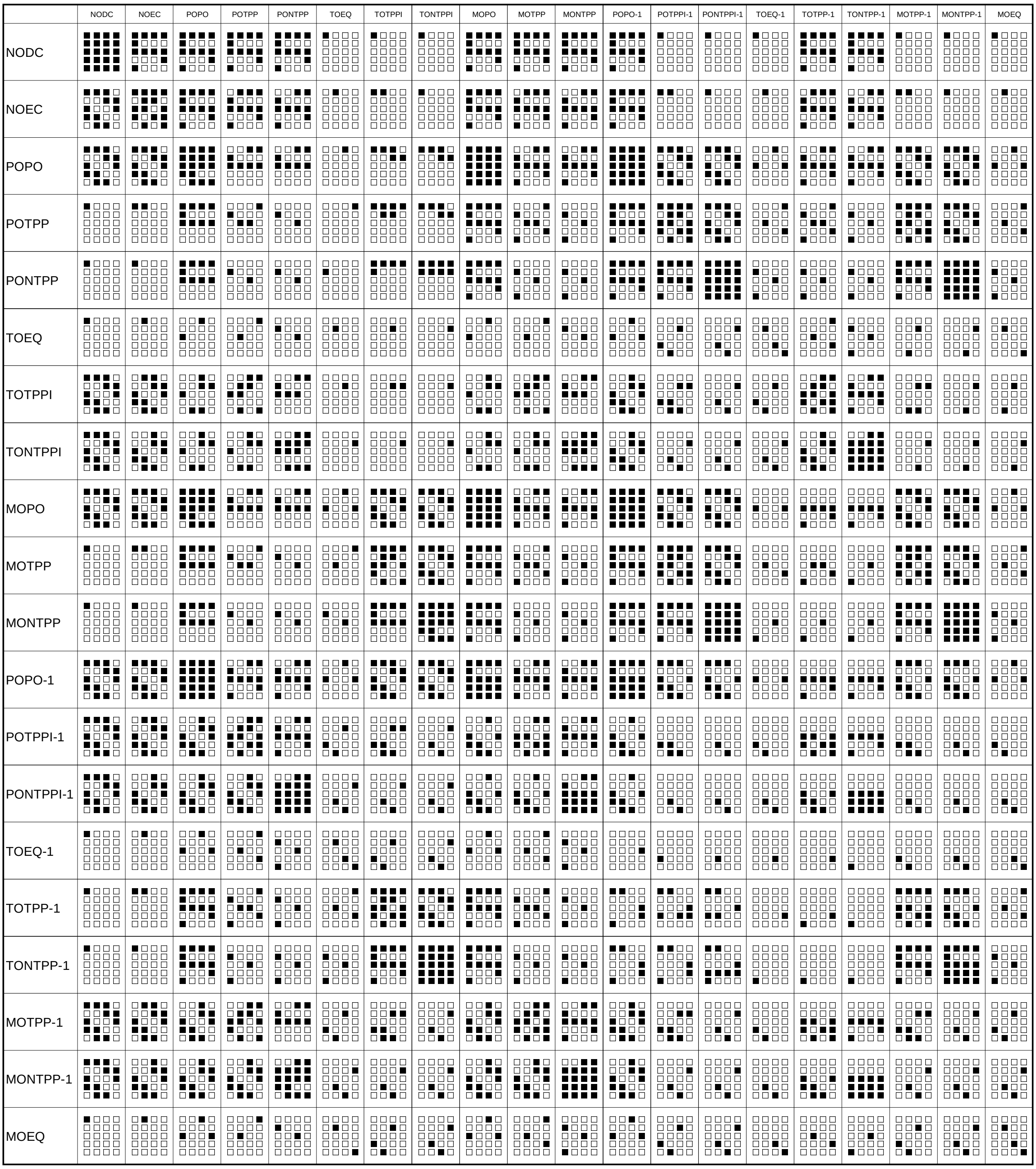


**Figure 6**: Full Composition Table for *ROC-20* - see Figure 1 for explanatory key