Information

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- Reference books:
  - B.P. Lathi, *Modern Digital and Analog Communication Systems*, Oxford University Press, 1998
  - S. Haykin, *Communication Systems*, Wiley, 2001
  - L.W. Couch II, *Digital and Analog Communication Systems*, Prentice-Hall, 2001

- Course material:
  - http://www.ee.ic.ac.uk/dward/

- Aims:

  The aim of this part of the course is to give you an understanding of how communication systems perform in the presence of noise.

- Objectives:

  By the end of the course you should be able to:
  - Compare the performance of various communication systems
  - Describe a suitable model for noise in communications
  - Determine the SNR performance of analog communication systems
  - Determine the probability of error for digital systems
  - Understand information theory and its significance in determining system performance

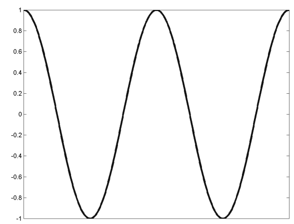
Lecture 1

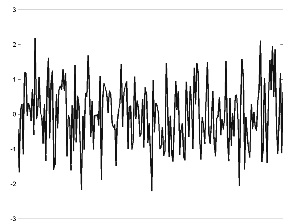
1. What is the course about, and how does it fit together
2. Some definitions (signals, power, bandwidth, phasors)

See Chapter 1 of notes

Definitions

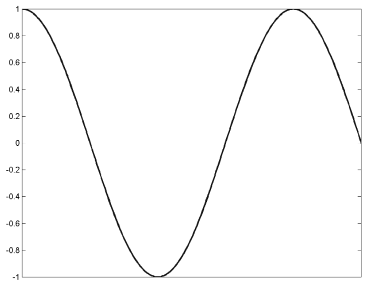
- **Signal**: a single-valued function of time that conveys information
- **Deterministic** signal: completely specified function of time
- **Random** signal: cannot be completely specified as function of time

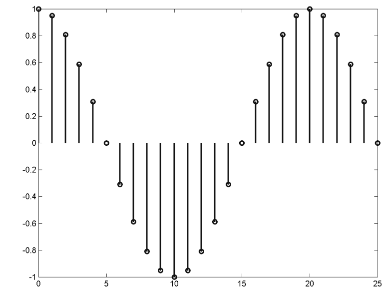




Definitions

- **Analog** signal: continuous function of time with continuous amplitude
- **Discrete-time** signal: only defined at discrete points in time, amplitude continuous
- **Digital** signal: discrete in both time and amplitude (e.g., PCM signals, see Chapter 4)





Definitions

- Instantaneous power:

$$p = \frac{|v(t)|^2}{R} = |i(t)|^2 R = |g(t)|^2$$

- Average power:

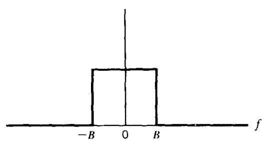
$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

For periodic signals, with period T_o (see Problem sheet 1):

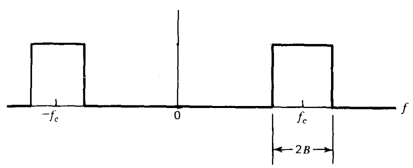
$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |g(t)|^2 \, dt$$

Definitions

- Bandwidth: extent of the significant spectral content of a signal for *positive* frequencies



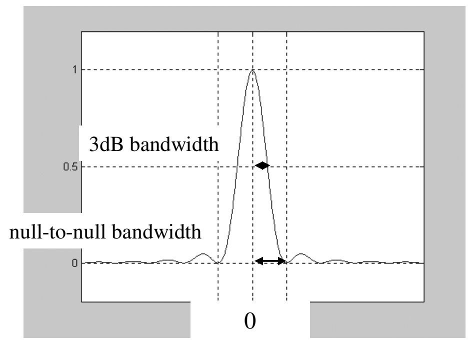
Baseband signal, B/W = B

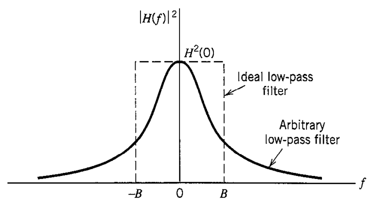


Bandpass signal, B/W = $2B$

Bandwidth

Magnitude-square spectrum



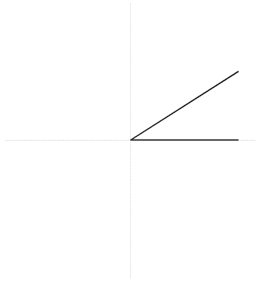


noise equivalent bandwidth

Phasors

- General sinusoid:

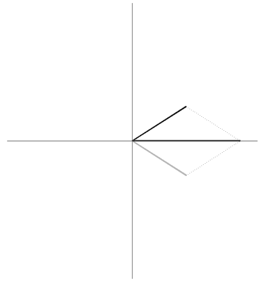
$$x(t) = A\cos(2\pi ft + \theta)$$
$$x(t) = \Re\left\{ A e^{j\theta} e^{j2\pi f t} \right\}$$



Phasors

- Alternative representation:

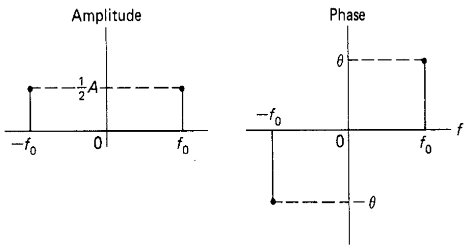
$$x(t) = A\cos(2\pi ft + \theta)$$
$$x(t) = \frac{A}{2} e^{j\theta} e^{j2\pi f t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi}$$



Phasors

$$x(t) = \frac{A}{2} e^{j\theta} e^{j2\pi f t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi f t}$$

- Anti-clockwise rotation (positive frequency): $\exp(j2\pi f t)$
- Clockwise rotation (negative frequency): $\exp(-j2\pi f t)$



Summary

1. The fundamental question: How do communications systems perform in the presence of noise?
2. Some definitions:
   - Signals
   - Average power $$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$
   - Bandwidth: significant spectral content for *positive* frequencies
   - Phasors – complex conjugate representation (negative frequency)

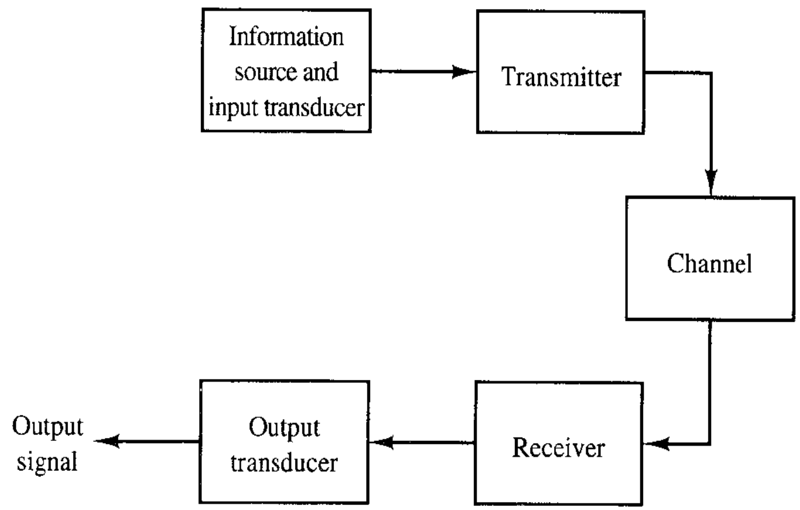
$$x(t) = \frac{A}{2} e^{j\theta} e^{j2\pi f t} + \frac{A}{2} e^{-j\theta} e^{-j2\pi f t}$$

Lecture 2

1. Model for noise
2. Autocorrelation and Power spectral density

See Chapter 2 of notes, sections 2.1, 2.2, 2.3

Sources of noise



Sources of noise

1. External noise
   - synthetic (e.g. other users)
   - atmospheric (e.g. lightning)
   - galactic (e.g. cosmic radiation)
2. Internal noise
   - shot noise
   - thermal noise

- Average power of thermal noise: $P = kTB$
- Effective noise temperature:

$$T_e = \frac{P}{kB}$$

Temperature of fictitious thermal noise source at i/p, that would be required to produce same noise power at o/p

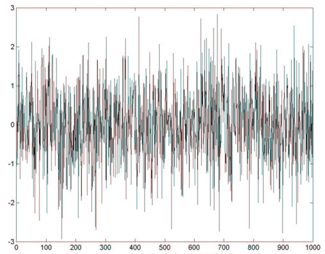
Example

- Consider an amplifier with 20 dB power gain and a bandwidth of $B = 20\,\text{MHz}$
- Assume the average thermal noise at its output is $P_o = 2.2\times10^{-11}$ W

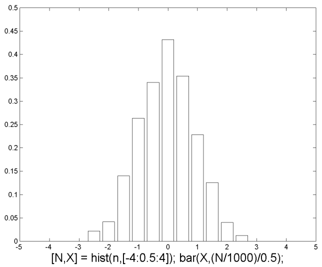
1. What is the amplifier's effective noise temperature?
2. What is the noise output if two of these amplifiers are cascaded?
3. How many stages can be used if the noise output must be less than 20 mW?

Gaussian noise

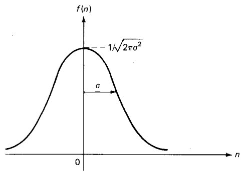
- **Gaussian noise**: amplitude of noise signal has a Gaussian probability density function (p.d.f.)



n=randn(1,1000)



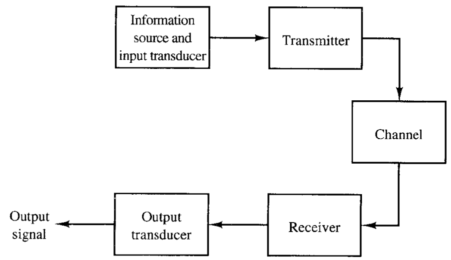
[N,X] = hist(n,[-4:0.5:4]); bar(X,(N/1000)/0.5);

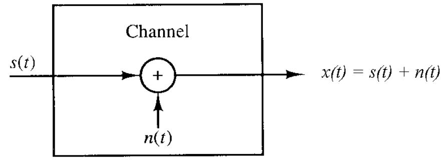


- *Central limit theorem* : sum of *n* independent random variables approaches Gaussian distribution as $n \to \infty$

Noise model

- Model for effect of noise is additive Gaussian noise channel:

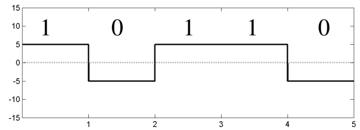




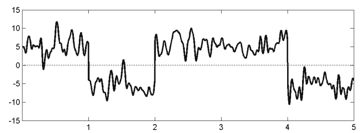
- For *n(t)* a random signal, need more info:
  - What happens to noise at receiver?
  - Statistical tools

Motivation

Data



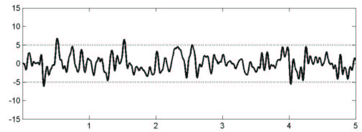
Data + noise



1
0

- How often is a decision error made?

Noise only

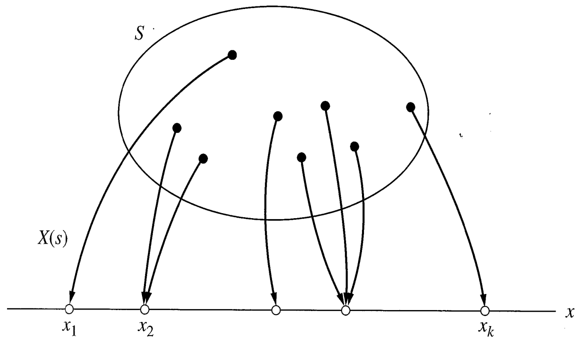


Error for 0

Error for 1

Random variable

- A random variable x is a rule that assigns a real number x_i to the ith sample point in the sample space



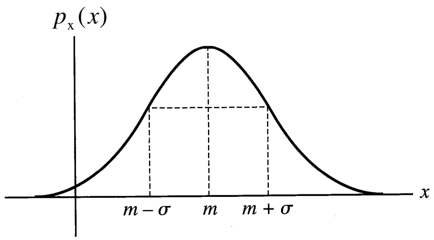
Probability density function

- Probability that the r.v. x is within a certain range is:

$$P(x_1 < \mathrm{x} < x_2) = \int_{x_1}^{x_2} p_{\mathrm{x}}(x)\,dx$$

- Example, Gaussian pdf:

$$p_{\mathrm{x}}(x) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-(x-m)^2/2\sigma^2}$$



Statistical averages

- Expectation of a r.v. is:

$$E\{\mathrm{x}\} = \int_{-\infty}^{\infty} x\, p_{\mathrm{x}}(x)\,dx$$

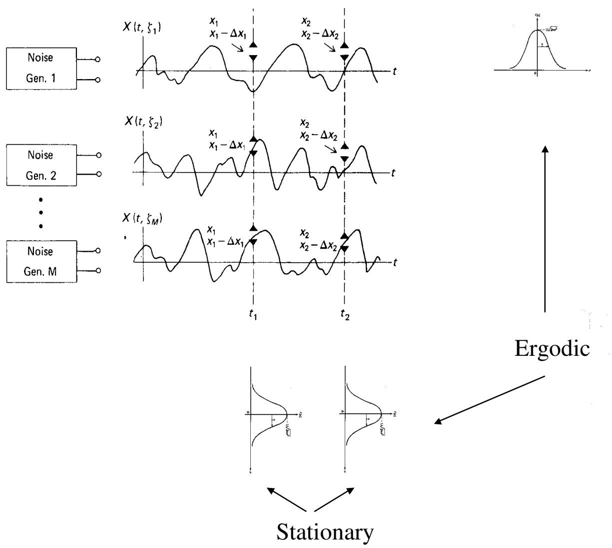
where $E\{\mathrm{x}\}$ is the *expectation operator*

- In general, if $\mathrm{y} = g(\mathrm{x})$

then $E\{\mathrm{y}\} = E\{g(\mathrm{x})\} = \int_{-\infty}^{\infty} g(x)\, p_{\mathrm{x}}(x)\,dx$

- For example, the mean square amplitude of a signal is the mean of the square of the amplitude, ie, $E\{\mathrm{x}^2\}$

Random process



Averages

- Time average:

$$\langle n(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} n(t)\,dt$$

- Ensemble average:

$$E\{n\} = \int_{-\infty}^{\infty} n\; p(n)\; dn$$

pdf random variable

- DC component: $E\{n(t)\} = \langle n(t)\rangle$

- Average power: $E\{n^2(t)\} = \langle n^2(t)\rangle = \sigma^2$ (zero-mean Gaussian process only)

Example

- Consider the signal: $x(t) = A\, e^{j(\omega_c t + \theta)}$

  where θ is a random variable, uniformly distributed over 0 to 2π

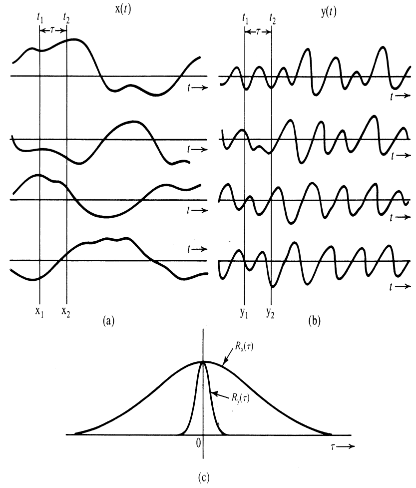
1. Calculate its average power using time averaging.
2. Calculate its average power (mean-square) using statistical averaging.

Autocorrelation

- How can one represent the spectrum of a random process?

Autocorrelation:

$$R_x(\tau) = E\{x(t)x(t+\tau)\}$$

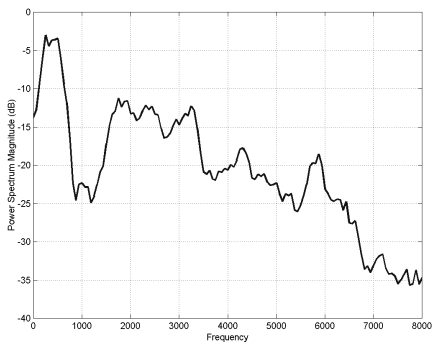


- NOTE: Average power is

$$P = E\{x^2(t)\}$$
$$= R_x(0)$$

Frequency content

Original speech → $H(f)$ → Filtered speech (smaller bandwidth)



Original LPF 4kHz LPF 1kHz

Power spectral density

- PSD measures distribution of power with frequency, units watts/Hz

Wiener-Khinchine theorem:

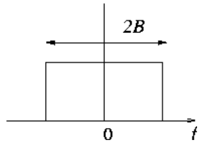
$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau}\, d\tau$$
$$= \mathrm{FT}\{R_x(\tau)\}$$

- Hence, $R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau}\, df$

- Average power:

$$P = R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$$

Power spectral density



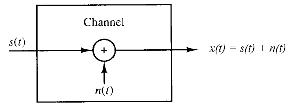
- Thermal noise: $P = kTB = \int_{-B}^{B} \frac{kT}{2}\, df$

- White noise: $S(f) = \frac{N_o}{2}$

PSD is same for all frequencies

Summary

- Additive Gaussian noise channel:



- Autocorrelation: $R_x(\tau) = E\{x(t)x(t+\tau)\}$
- Power spectral density:

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

- White noise: $S(f) = \frac{N_o}{2}$
- Expectation operator:

pdf random variable

$$E\{g(x)\} = \int_{-\infty}^{\infty} [g(x)]\, p_x(x)\, dx$$

Lecture 3

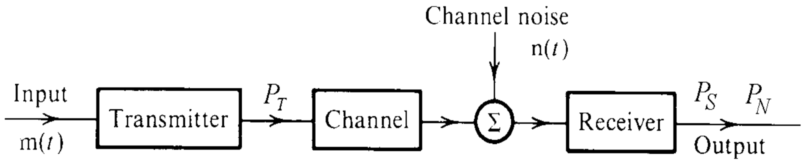
- Representation of band-limited noise
  - Why band-limited noise?

  (See Chapter 2, section 2.4)

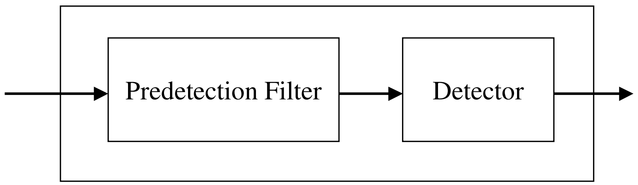
- Noise in an analog baseband system

  (See Chapter 3, sections 3.1, 3.2)

Analog communication system



Receiver



- Predetection filter:
  - removes out-of-band noise
  - has a bandwidth matched to the transmission bandwidth

Bandlimited Noise

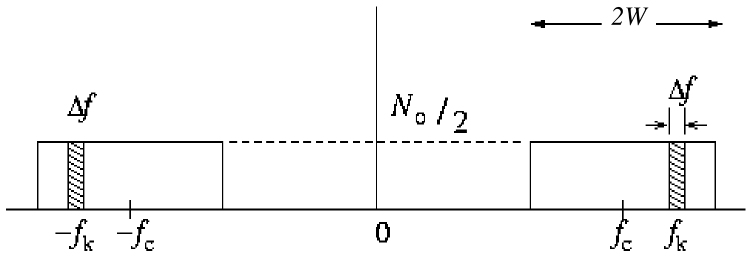
- For any bandpass (i.e., modulated) system, the predetection noise will be bandlimited
- Bandpass noise signal can be expressed in terms of two baseband waveforms

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

bandpass baseband carrier baseband carrier

- PSD of $n(t)$ is centred about f_c (and $-f_c$)
- PSDs of $n_c(t)$ and $n_s(t)$ are centred about 0 Hz

PSD of *n(t)*



- In slice shown (for Δf small):

$$n_k(t) = a_k \cos(2\pi f_k t + \theta_k)$$

$$n_k(t) = a_k \cos(\omega_k t + \theta_k)$$

let $\omega_k = (\omega_k - \omega_c) + \omega_c$

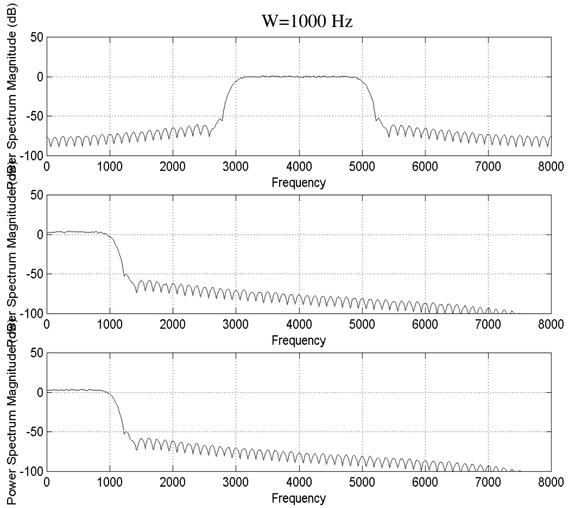
$$n_k(t) = a_k \cos\big[\underbrace{(\omega_k - \omega_c)t + \theta_k}_{\text{A}} + \underbrace{\omega_c t}_{\text{B}}\big]$$

use $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$n_k(t) = \overbrace{a_k \cos[(\omega_k - \omega_c)t + \theta_k]}^{n_c(t)\ \text{term}} \cos(\omega_c t)$$

$$- \underbrace{a_k \sin[(\omega_k - \omega_c)t + \theta_k]}_{n_s(t)\ \text{term}} \sin(\omega_c t)$$

Example – frequency

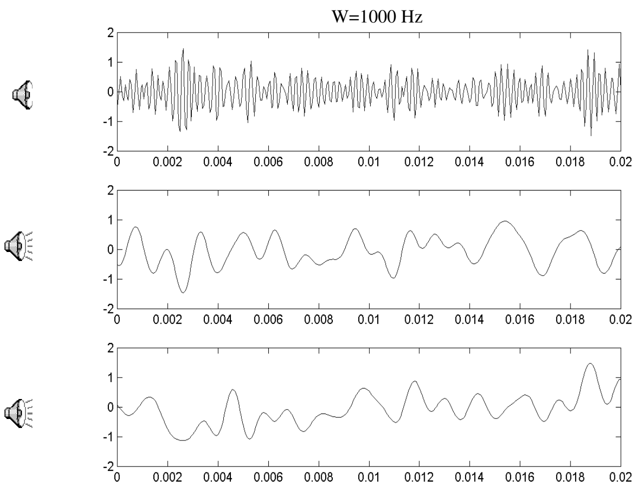


n(t)

$n_c(t)$

$n_s(t)$

Example – time

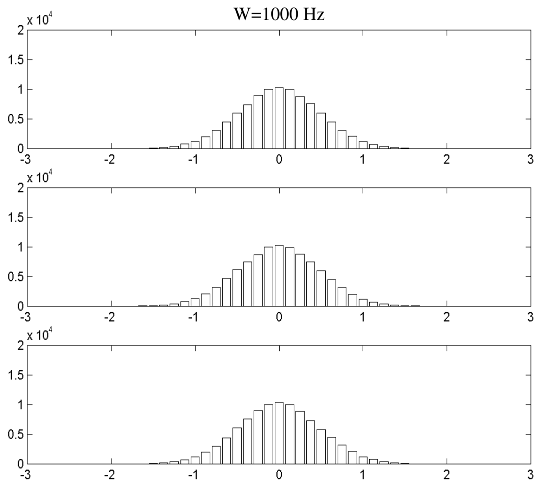


n(t)

$n_c(t)$

$n_s(t)$

Example – histogram



$n(t)$

$n_c(t)$

$n_s(t)$

Probability density functions

$$n(t) = \sum_k a_k \cos(\omega_k t + \theta_k)$$
$$n_c(t) = \sum_k a_k \cos\left[(\omega_k - \omega_c)t + \theta_k\right]$$
$$n_s(t) = \sum_k a_k \sin\left[(\omega_k - \omega_c)t + \theta_k\right]$$

- Each waveform is Gaussian distributed
  - Central limit theorem
- Mean of each waveform is 0

Average power

$$n(t) = \sum_k a_k \cos(\omega_k t + \theta_k)$$

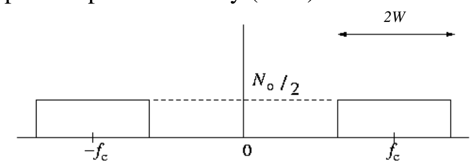
- Power in $a_k \cos(\omega t+\theta)$ is $E\{a_k^2\}/2$ (see Example 1.1, or study group sheet 2, Q1)
- Average power in $n(t)$ is: $P_n = \sum_k \frac{E\{a_k^{\ 2}\}}{2}$

$$n_c(t) = \sum_k a_k \cos\left((\omega_k - \omega_c)t + \theta_k\right)$$
$$n_s(t) = \sum_k a_k \sin\left((\omega_k - \omega_c)t + \theta_k\right)$$

- Average power in $n_c(t)$ and $n_s(t)$ is: $P_{n_c} = \sum_k \frac{E\{a_k^{\ 2}\}}{2}, \quad P_{n_s} = \sum_k \frac{E\{a_k^{\ 2}\}}{2}$
- $n(t)$, $n_c(t)$ and $n_s(t)$ all have same average power!

Average power

- What is the average power in $n(t)$?
- Find using the power spectral density (PSD):



- From Lecture 2: $P = \int_{-\infty}^{\infty} S(f)\, df$

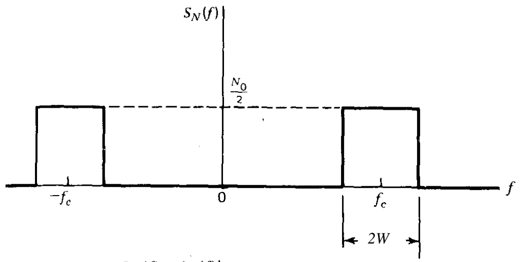
$$= 2\int_{f_c-W}^{f_c+W} \frac{N_o}{2}\, df = 2\frac{N_o}{2}(W - {}^{-}W) = 2N_o W$$

(one for positive freqs, one for negative)

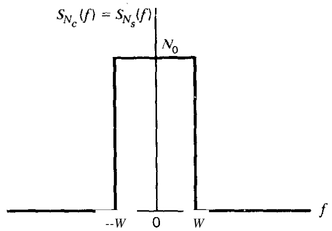
- **Average power in $n(t)$, $n_c(t)$ and $n_s(t)$ is: $2N_oW$**

Power spectral densities

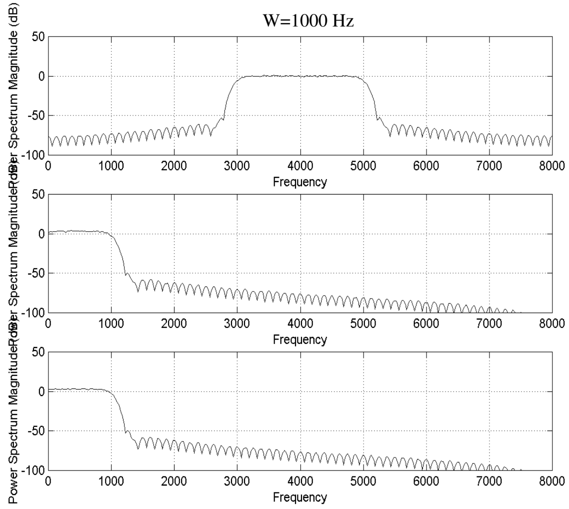
PSD of *n(t)*:



PSD of $n_c(t)$ and $n_s(t)$:



Example – power spectral densities

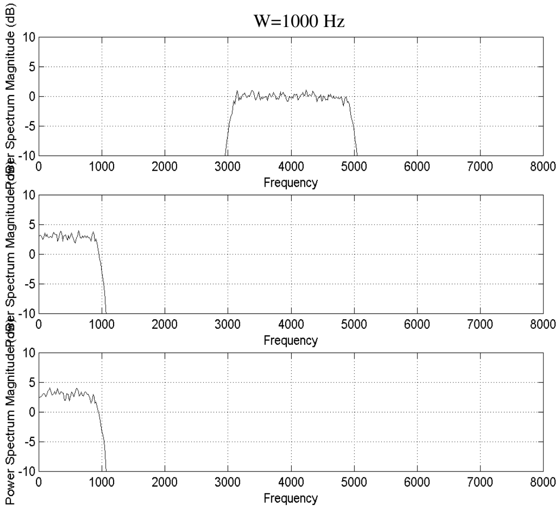


$n(t)$

$n_c(t)$

$n_s(t)$

Example - zoomed



$n(t)$

$n_c(t)$

$n_s(t)$

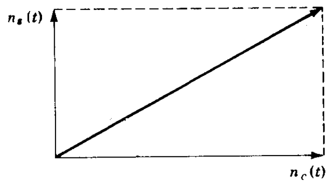
Phasor representation

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

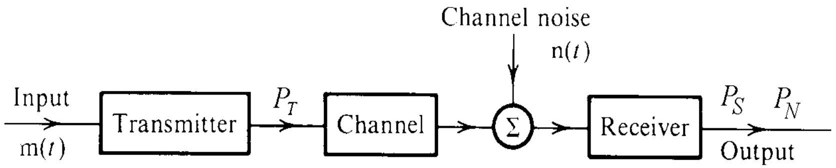
let $g(t) = n_c(t) + jn_s(t)$

$$g(t)\,e^{j\omega_c t} = n_c(t)\cos\omega_c t + jn_c(t)\sin\omega_c t + jn_s(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$

so $n(t) = \Re\left\{g(t)\,e^{j\omega_c t}\right\}$



Analog communication system

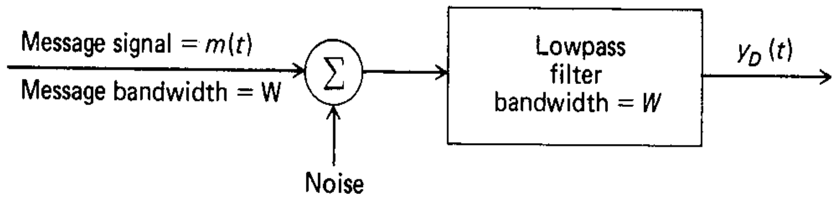


- Performance measure:

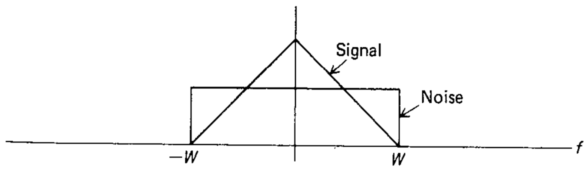
$$SNR_o = \frac{\text{average message power at receiver output}}{\text{average noise power at receiver output}}$$

- Compare systems with same transmitted power

Baseband system



Baseband SNR



- Transmitted (message) power is: P_T
- Noise power is:

$$P_N = \int_{-W}^{W} \frac{N_o}{2}\, df = N_o W$$

- **SNR at receiver output:**

$$SNR_{\text{base}} = \frac{P_T}{N_o W}$$

Summary

- Baseband SNR:

$$SNR_{\text{base}} = \frac{P_T}{N_o W}$$

- Bandpass noise representation:

$$n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$$

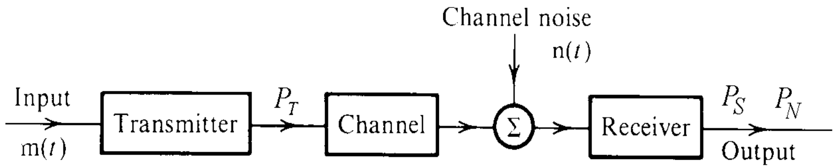
- All waveforms have same:
  - Probability density function (zero-mean Gaussian)
  - Average power
- $n_c(t)$ and $n_s(t)$ have same power spectral density

Lecture 4

- Noise (AWGN) in AM systems:
  - DSB-SC
  - AM, synchronous detection
  - AM, envelope detection

(See Chapter 3, section 3.3)

Analog communication system



- Performance measure:

$$SNR_o = \frac{\text{average message power at receiver output}}{\text{average noise power at receiver output}}$$

- Compare systems with same transmitted power

Amplitude modulation

- Modulated signal:

$$s(t)_{\mathrm{AM}} = [A_c + m(t)]\cos\omega_c t$$

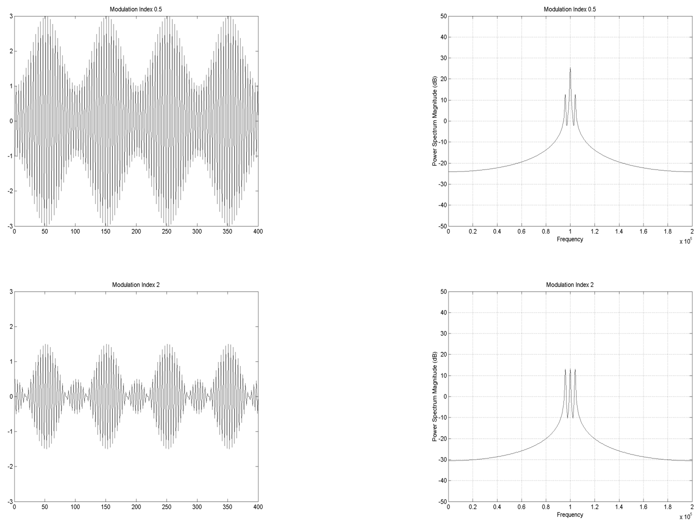
m(t) is message signal

- Modulation index:

$$\mu = \frac{m_p}{A_c}$$

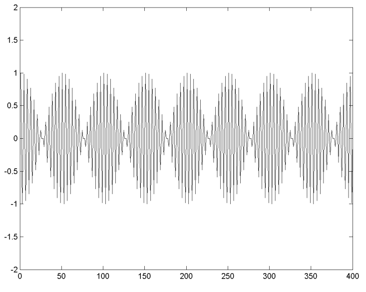
m_p is peak amplitude of message

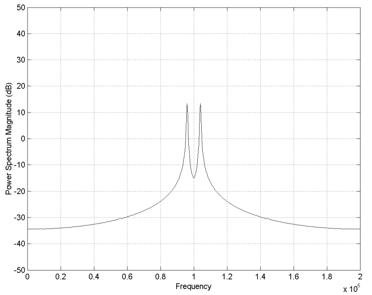
Amplitude modulation



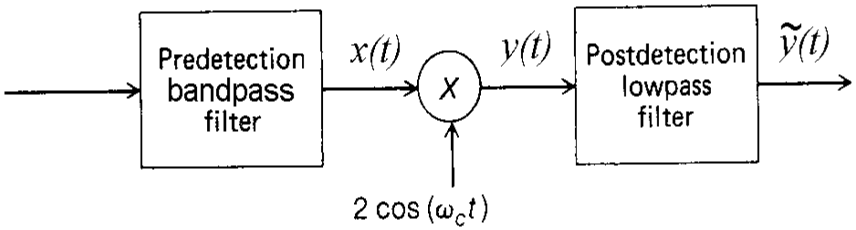
DSB-SC

$$s(t)_{\text{DSB-SC}} = A_c m(t) \cos \omega_c t$$





Synchronous detection



- Signal after multiplier:

$$\begin{aligned} y(t)_{AM} &= [A_c + m(t)] \cos \omega_c t \times 2 \cos \omega_c t \\ &= [A_c + m(t)](1 + \cos 2\omega_c t) \end{aligned}$$

$$y(t)_{DSB-SC} = A_c m(t)(1 + \cos 2\omega_c t)$$

Noise in DSB-SC

- Transmitted signal:

$$s(t)_{\text{DSB-SC}} = A_c m(t) \cos \omega_c t$$

- Predetection signal:

$$x(t) = \underbrace{A_c m(t) \cos \omega_c t}_{\text{Transmitted signal}} + \underbrace{n_c(t) \cos \omega_c t - n_s(t) \sin \omega_c t}_{\text{Bandlimited noise}}$$

- Receiver output (after LPF):

$$y(t) = A_c m(t) + n_c(t)$$

SNR of DSB-SC

- Output signal power:

$$P_s = \left\langle (A_c m(t))^2 \right\rangle = A_c^2 \ P$$

power of message

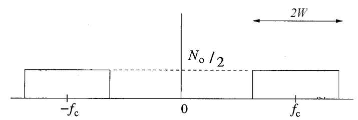
- Output noise power:

$$P_N = \int_{-\infty}^{\infty} \text{PSD } df = \int_{-W}^{W} N_o \ df = 2 N_o W$$

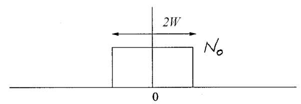
PSD of $n_c(t)$

- Output SNR:

$$SNR_{\text{DSB-SC}} = \frac{A_c^2 P}{2 N_o W}$$



PSD of bandpass noise *n(t)*



PSD of baseband noise $n_c(t)$

SNR of DSB-SC

- Transmitted power:

$$P_T = \left\langle (A_c m(t)\cos\omega_c t)^2 \right\rangle = \frac{A_c^2 P}{2}$$

- Output SNR:

$$SNR_{\text{DSB-SC}} = \frac{P_T}{N_o W} = SNR_{\text{base}}$$

- **DSB-SC has no performance advantage over baseband**

Noise in AM (synch. detector)

- Predetection signal:

$$x(t) = \underbrace{\left[A_c + m(t)\right]\cos\omega_c t}_{\text{Transmitted signal}} + \underbrace{n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t}_{\text{Bandlimited noise}}$$

- Receiver output:

$$y(t) = \cancel{A_c} + m(t) + n_c(t)$$

- Output signal power:

$$P_S = \left\langle m^2(t) \right\rangle = P$$

- Output noise power:

$$P_N = 2N_o W$$

- Output SNR:

$$SNR_{AM} = \frac{P}{2N_o W}$$

Noise in AM (synch. detector)

- Transmitted signal:

$$s(t)_{\text{AM}} = \left[A_c + m(t)\right]\cos\omega_c t$$

- Transmitted power:

$$P_T = \frac{A_c^2}{2} + \frac{P}{2}$$

- Output SNR:

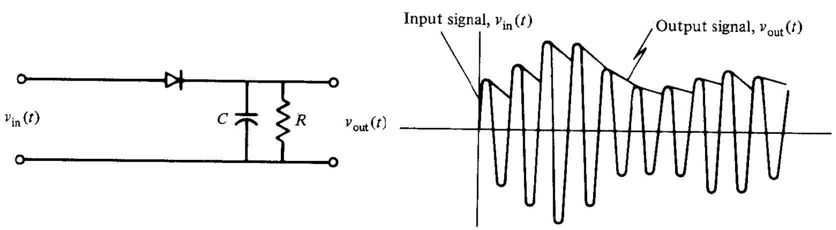
$$SNR_{AM} = \frac{P_T}{N_o W}\frac{P}{A_c^2 + P} = \frac{P}{A_c^2 + P} SNR_{\text{base}}$$

- The performance of **AM is always worse than baseband**

Noise in AM, envelope detector

- Predetection signal:

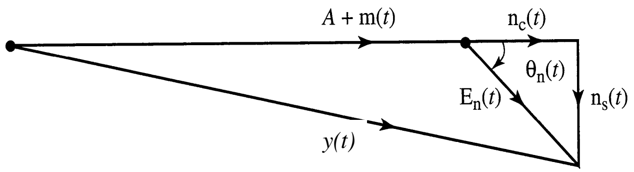
$$x(t) = \underbrace{\left[A_c + m(t)\right]\cos\omega_c t}_{\text{Transmitted signal}} + \underbrace{n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t}_{\text{Bandlimited noise}}$$



Noise in AM, envelope detector

- Receiver output:

$$y(t) = \text{envelope of } x(t)$$
$$= \sqrt{[A_c + m(t) + n_c(t)]^2 + n_s^2(t)}$$



Noise in AM, envelope detector

- Small noise case:

$$y(t) \approx A_c + m(t) + n_c(t)$$
$$= \text{output of synchronous detector}$$

- Large noise case:

$$y(t) \approx E_n(t) + [A_c + m(t)]\cos\theta_n(t)$$

- Envelope detector has a **threshold effect**
- Not really a problem in practice

Example

- An unmodulated carrier (of amplitude A_c and frequency f_c) and bandlimited white noise are summed and then passed through an ideal envelope detector.
- Assume the noise spectral density to be of height $N_o/2$ and bandwidth $2W$, centred about the carrier frequency.
- Assume the input carrier-to-noise ratio is high.

1. Calculate the carrier-to-noise ratio at the output of the envelope detector, and compare it with the carrier-to-noise ratio at the detector input.

Summary

- Synchronous detector:

$$SNR_{\text{DSB-SC}} = SNR_{\text{base}}$$

$$SNR_{AM} = \frac{P}{A_c^2 + P} SNR_{\text{base}}$$

- Envelope detector:
  - threshold effect
  - for small noise, performance is same as synchronous detector

Lecture 5

- Noise in FM systems
  - pre-emphasis and de-emphasis

(See section 3.4)

- Comparison of analog systems

(See section 3.5)

Frequency modulation

- FM waveform:

$$s(t)_{\mathrm{FM}} = A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau\right)$$
$$= A_c \cos\theta(t)$$

- $\theta(t)$ is instantaneous phase
- Instantaneous frequency:

$$f_i = \frac{1}{2\pi}\frac{d\theta(t)}{dt}$$
$$= f_c + k_f m(t)$$

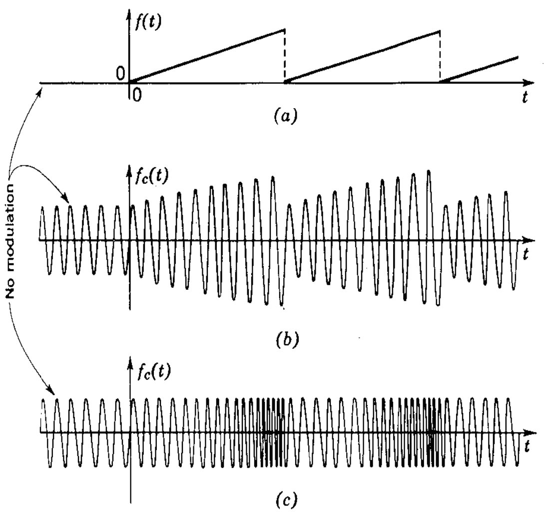
- Frequency is proportional to message signal

FM waveforms

Message:

AM:

FM:



FM frequency deviation

- Instantaneous frequency varies between f_c-$k_f\, m_p$ and f_c+$k_f\, m_p$, where m_p is peak message amplitude
- Frequency deviation (max. departure of carrier wave from f_c):

$$\Delta f = k_f m_p$$

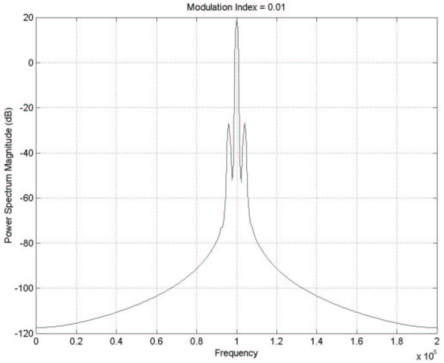
- Deviation ratio:

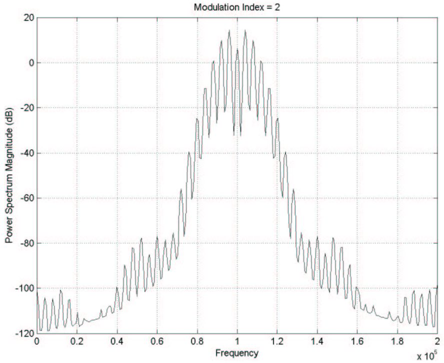
$$\beta = \frac{\Delta f}{W}$$

where W is bandwidth of message signal

- Commercial FM uses: Δf=75 kHz and W=15 kHz

Bandwidth considerations

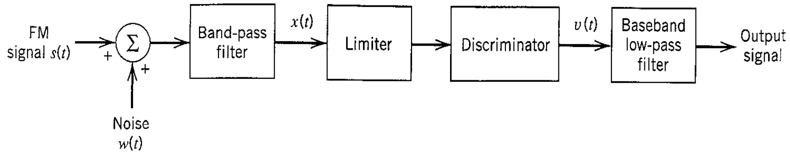




- Carson's rule:

$$B_T \approx 2W(\beta+1) = 2(\Delta f + W)$$

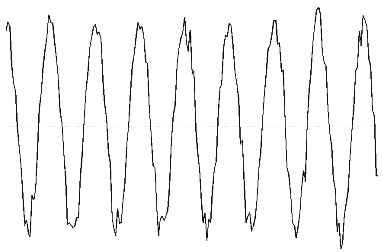
FM receiver



- Discriminator: output is proportional to deviation of instantaneous frequency away from carrier frequency

Noise in FM versus AM

- AM:
  - Amplitude of modulated signal carries message
  - Noise adds directly to modulated signal
  - Performance no better than baseband
- FM:
  - Frequency of modulated signal carries message
  - Zero crossings of modulated signal important
  - Effect of noise should be less than in AM



Noise in FM

- Predetection signal:

$$x(t) = \overbrace{A_c \cos\left(2\pi f_c t + \phi(t)\right)}^{\text{Transmitted signal}} + \overbrace{n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)}^{\text{Bandlimited noise}}$$

$$\text{where } \phi(t) = 2\pi\, k_f \int_{-\infty}^{t} m(\tau)\,d\tau$$

- If carrier power is much larger than noise power:
  1. Noise does not affect signal power at output
  2. Message does not affect noise power at output

Assumptions

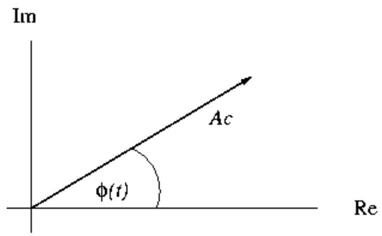
1. Noise does not affect signal power at output

- Signal component at receiver:

$$x_s(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right)$$

- Instantaneous frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = k_f m(t)$$



- Output signal power:

$$P_S = k_f^2 P$$

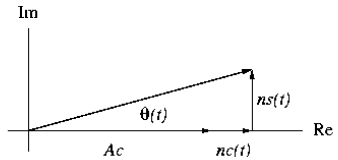
power of message signal

Assumptions

2. Signal does not affect noise at output

- Message-free component at receiver:

$$x_n(t) = A_c \cos\left(2\pi f_c t\right) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$



- Instantaneous frequency:

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\tan^{-1}\left\{\frac{n_s(t)}{A_c + n_c(t)}\right\} \approx \frac{1}{2\pi}\frac{d}{dt}\left(\frac{n_s(t)}{A_c}\right)$$

- We know the PSD of $n_s(t)$, but what about its derivative??

- Discriminator output:

$$f_i(t) \approx \frac{1}{2\pi A_c}\frac{dn_s(t)}{dt}$$

$$n_s(t) \longrightarrow \boxed{\frac{1}{2\pi A_c}\frac{d}{dt}} \longrightarrow f_i(t)$$

- Fourier theory property:

$$x(t) \Leftrightarrow X(f)$$

$$\frac{dx(t)}{dt} \Leftrightarrow j2\pi f X(f)$$

$$N_s(f) \longrightarrow \boxed{\frac{1}{2\pi A_c} j2\pi f} \longrightarrow F_i(f)$$

- PSD property:

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = H(f)X(f)$$

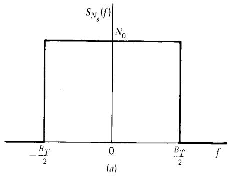
$$S_X(f) \qquad S_Y(f) = |H(f)|^2 S_X(f)$$

- PSD of discriminator output:

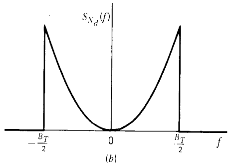
$$N_s(f) \longrightarrow \boxed{j\frac{f}{A_c}} \longrightarrow F_i(f)$$

$$S_N(f) \qquad S_F(f) = \frac{|f|^2}{A_c^2} S_N(f)$$

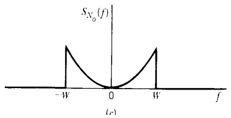
PSD of $n_s(t)$



PSD of $\frac{1}{2\pi A}\frac{dn_s(t)}{dt}$



PSD after LPF



- PSD of LPF noise term:

$$\text{PSD of } \frac{1}{2\pi A_c}\frac{dn_s(t)}{dt} = \frac{|f|^2}{A_c^2} N_o, \quad |f| < W$$

- Average power of noise at output:

$$P_N = \int_{-W}^{W} \frac{|f|^2}{A_c^2} N_o \, df = \frac{2N_o}{A_c^2}\left[\frac{f^3}{3}\right]_0^W = \frac{2N_o W^3}{3A_c^2}$$

- Increasing the carrier power has a **noise quieting** effect

SNR of FM

- SNR at output:

$$SNR_o = \frac{3A_c^2 k_f^2 P}{2N_o W^3}$$

- Transmitted power:

$$P_T = \left\langle \left(A_c \cos[\omega_c t + \phi(t)]\right)^2 \right\rangle = \frac{A_c^2}{2}$$

- SNR at output:

$$SNR_{\text{FM}} = \frac{3k_f^2 P}{W^2} SNR_{\text{base}} = \frac{3\beta^2 P}{m_p} SNR_{\text{base}}$$

Threshold effect in FM

- SNR_{FM} is valid when the **predetection** SNR > 10
- Predetection signal is:

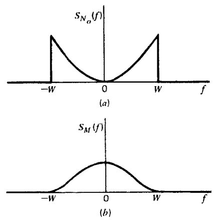
$$x(t) = A_c \cos\left(2\pi f_c t + \phi(t)\right) + n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

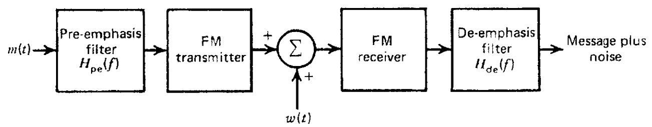
- Predetection SNR is: $SNR_{pre} = \frac{A_c^2}{2N_o B_T}$

- Threshold point is: $\frac{A_c^2}{4N_o W(\beta+1)} > 10$

- **Cannot arbitrarily increase SNR_{FM} by increasing β**

Pre-emphasis and De-emphasis





- Can improve output SNR by about 13 dB

Example

- The improvement in output SNR afforded by using pre-emphasis and de-emphasis in FM is defined by:

$$I = \frac{SNR \text{ with pre-/de-emphasis}}{SNR \text{ without pre-/de-emphasis}} = \frac{\text{average output noise power without pre-/de-emphasis}}{\text{average output noise power with pre-/de-emphasis}}$$

- If $H_{de}(f)$ is the transfer function of the de-emphasis filter, find an expression for the improvement, I.

Analog system performance

- Parameters:
  - Single-tone message $m(t) = \cos(2\pi f_m t)$
  - Message bandwidth $W = f_m$
  - AM system $\mu = 1$
  - FM system $\beta = 5$

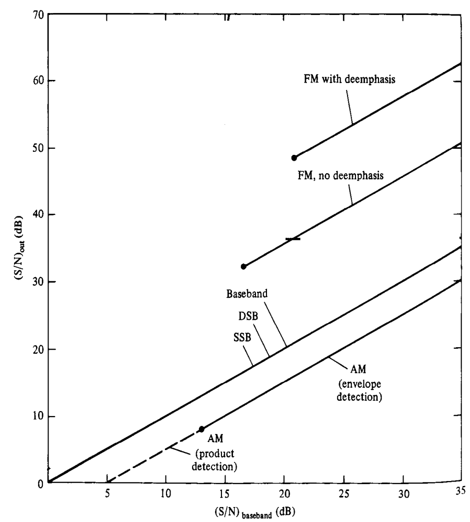
- Performance:

$$SNR_{DSB-SC} = SNR_{base}$$
$$SNR_{AM} = \tfrac{1}{3} SNR_{base}$$
$$SNR_{FM} = \tfrac{75}{2} SNR_{base}$$

- Bandwidth:

$$B_{DSB-SC} = 2W$$
$$B_{AM} = 2W$$
$$B_{FM} = 12W$$

Analog system performance



Summary

- Noise in FM:
  - Increasing carrier power reduces noise at receiver output
  - Has threshold effect
  - Pre-emphasis

- Comparison of analog modulation schemes:
  - AM worse than baseband
  - DSB/SSB same as baseband
  - FM better than baseband

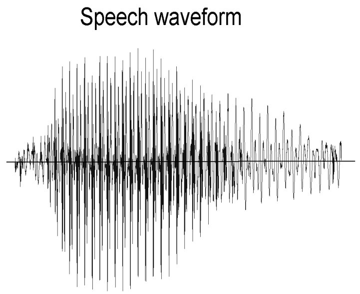
Lecture 6

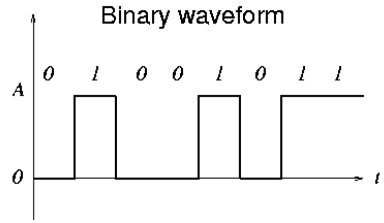
- Digital communication systems
  - Digital vs Analog communications
  - Pulse Code Modulation

(See sections 4.1, 4.2 and 4.3)

Digital vs Analog

- Analog message:
- Digital message:



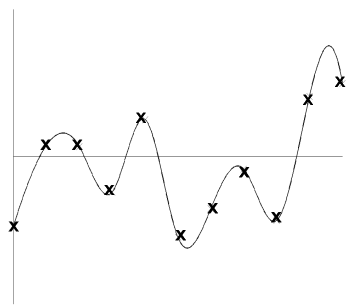


Digital vs Analog

- Analog:
  - **Recreate** waveform accurately
  - Performance criterion is **SNR at receiver output**
- Digital:
  - **Decide** which symbol was sent
  - Performance criterion is **probability of receiver making a decision error**

- Advantages of digital:
  1. Digital signals are more immune to noise
  2. Repeaters can re-transmit a noise-free signal

Sampling: discrete in time



Nyquist Sampling Theorem:
A signal whose bandwidth is limited to W Hz can be reconstructed exactly from its samples taken uniformly at a rate of $R>2W$ Hz.

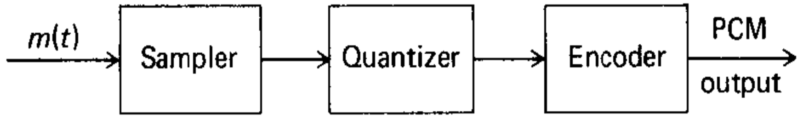
Maximum information rate

- How many bits can be transferred over a channel of bandwidth B Hz (ignoring noise)?
- Signal with a bandwidth of B Hz is not distorted over this channel
- Signal with a bandwidth of B Hz requires samples taken at $2B$ Hz
- Can transmit:

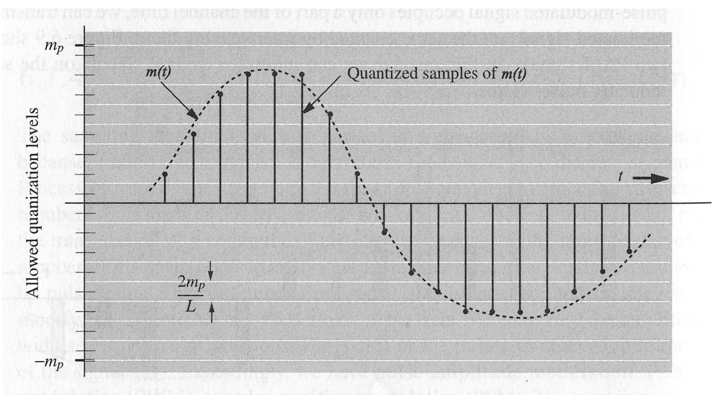
  2 bits of information per second per Hz

Pulse-code modulation

- Represent an analog waveform in digital form

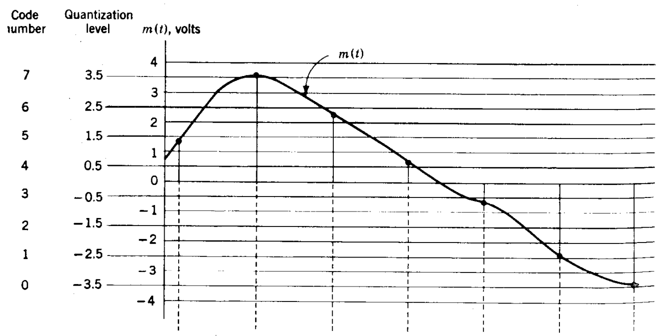


Quantization: discrete in amplitude



- Round amplitude of each sample to nearest one of a finite number of levels

Encode



Sample value	1.3	3.6	2.3	0.7	−0.7	−2.4	−3.4
Nearest quantization level	1.5	3.5	2.5	0.5	−0.5	−2.5	−3.5
Code number	5	7	6	4	3	1	0
Binary representation	101	111	110	100	011	001	000

- Assign each quantization level a code

Sampling vs Quantization

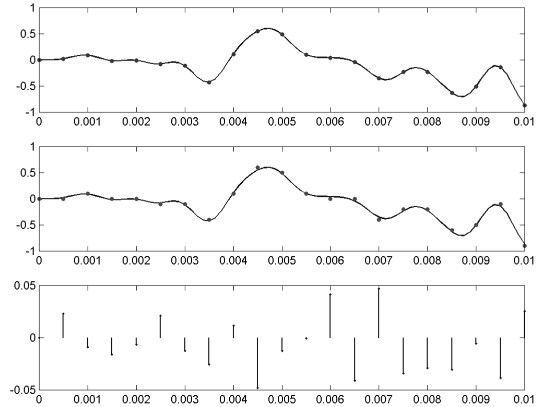
- Sampling:
  - Non-destructive if $f_s > 2W$
  - Can reconstruct analog waveform exactly by using a low-pass filter
- Quantization:
  - Destructive
  - Once signal has been rounded off it can never be reconstructed exactly

Quantization noise

Sampled signal

Quantized signal
(step size of 0.1)

Quantization error



Quantization noise

- Let Δ be the separation between quantization levels

$$\Delta = \frac{2m_p}{L}$$

where $L=2^n$ is the no. of quantization levels
m_p is peak **allowed** signal amplitude

- Round-off effect of quantizer ensures that $|q| < \Delta/2$, where q is a random variable representing the quantization error

- Assume q is zero mean with uniform pdf, so mean square error is:

$$E\{q^2\} = \int_{-\infty}^{\infty} q^2\, p(q)\, dq$$
$$= \int_{-\Delta/2}^{\Delta/2} q^2 \tfrac{1}{\Delta}\, dq = \frac{\Delta^2}{12}$$

Quantization noise

- Let message power be P
- Noise power is: $P_N = E\{q^2\}$ (since zero mean)

$$= \frac{\Delta^2}{12} = \frac{\left(2m_p/L\right)^2}{12} = \frac{m_p^2}{3 \times 2^{2n}}$$

- Output SNR of quantizer:

$$SNR_Q = \frac{P_S}{P_N} = \frac{3P}{m_p^2} \times 2^{2n}$$

or in dB:

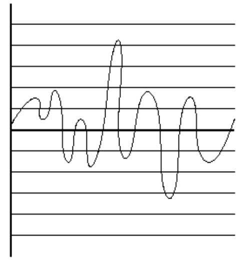
$$SNR_Q = 6.02n + 10\log_{10}\left(\frac{3P}{m_p^2}\right) \text{ dB}$$

Bandwidth of PCM

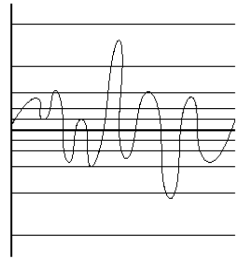
- Each message sample requires n bits
- If message has bandwidth W Hz, then PCM contains $2nW$ bits per second
- Bandwidth required is: $B_T = nW$
- SNR can be written: $SNR_Q = \frac{3P}{m_p^2} \times 2^{2B_T/W}$
- Small increase in bandwidth yields a large increase in SNR

Nonuniform quantization

- For audio signals (e.g. speech), small signal amplitudes occur more often than large signal amplitudes
- Better to have closely spaced quantization levels at low signal amplitudes, widely spaced levels at large signal amplitudes
- Quantizer has better resolution at low amplitudes (where signal spends more time)

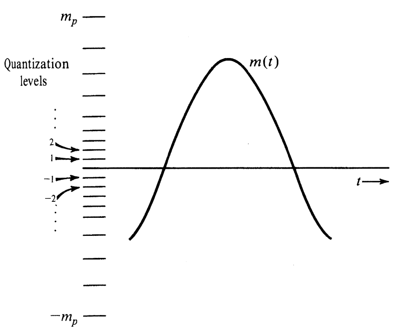


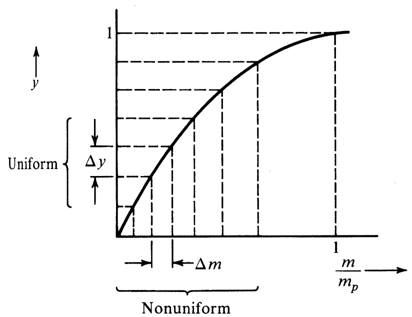
Uniform quantization



Non-uniform quantization

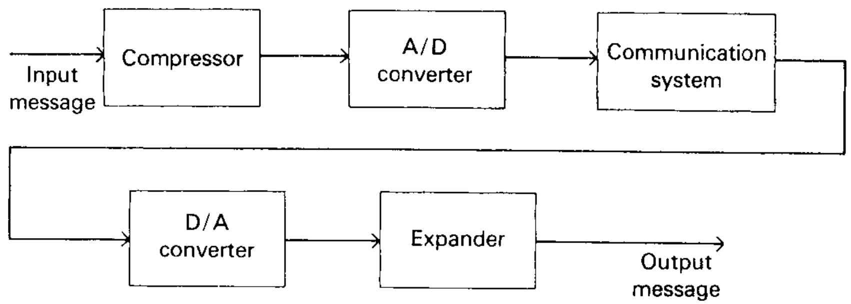
Nonuniform quantization





- Uniform quantizer is easier to implement that nonlinear
- **Comp**ress signal first, then use uniform quantizer, then exp**and** signal (i.e., compand)

Companding



Summary

- Analog communications system:
  - Receiver must recreate transmitted waveform
  - Performance measure is signal-to-noise ratio
- Digital communications system:
  - Receiver must decide which symbol was transmitted
  - Performance measure is probability of error

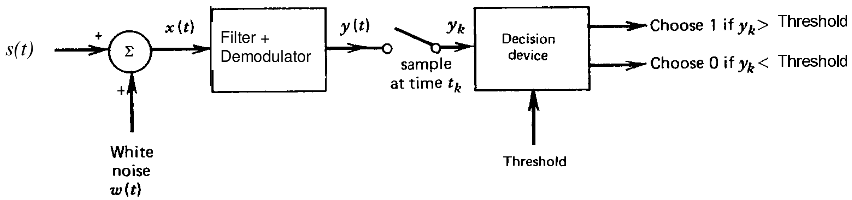
- Pulse-code modulation:
  - Scheme to represent analog signal in digital format
  - Sample, quantize, encode
  - Companding (nonuniform quantization)

Lecture 7

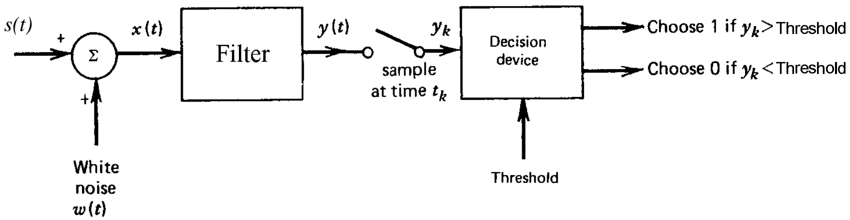
- Performance of digital systems in noise:
  - Baseband
  - ASK
  - PSK, FSK
  - Compare all schemes

(See sections 4.4, 4.5)

Digital receiver

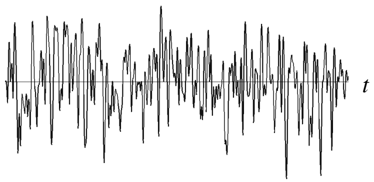


Baseband digital system

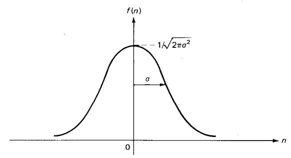


Gaussian noise, probability

Noise waveform, *n(t)*



Probability density function, *f(n)*



$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(n-m)^2}{2\sigma^2}\right)$$

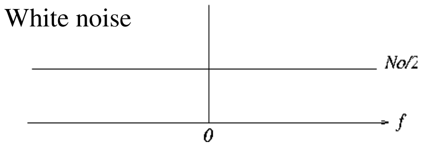
$$= \mathbf{N}(m, \sigma^2)$$

Normal distribution
mean, *m*
variance, σ^2

$$\text{prob}(a < n < b) = \int_a^b p(n)\, dn$$

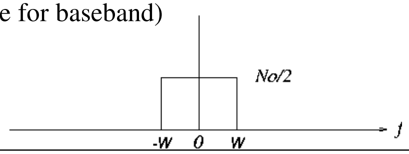
Gaussian noise, spectrum

White noise



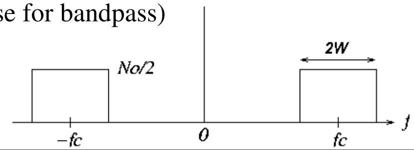
NOTE: For **zero-mean** noise,
variance ≡ average power
i.e., $\sigma^2 = P$

LPF (use for baseband)



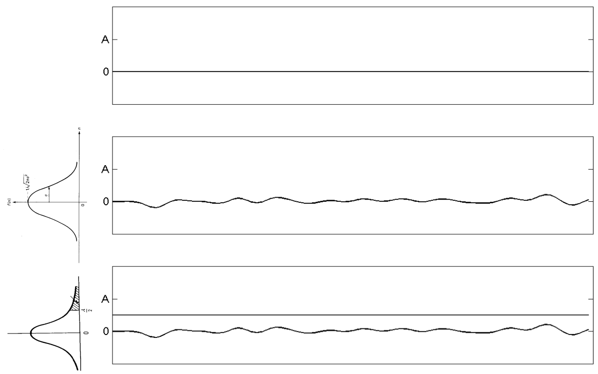
$$P = \int_{-W}^{W} \frac{N_o}{2}\, df = N_o W$$

BPF (use for bandpass)



$$P = \int_{f_c - W}^{f_c + W} \frac{N_o}{2}\, df + \int_{-f_c - W}^{-f_c + W} \frac{N_o}{2}\, df$$
$$= 2 N_o W$$

Baseband system – "0" transmitted



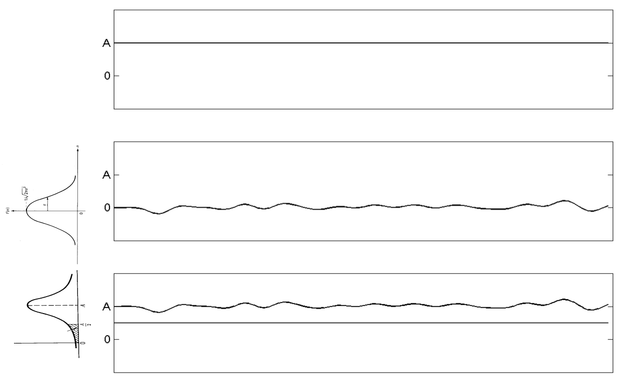
Transmitted signal
$s_0(t)$

Noise signal
$n(t)$

Received signal
$y_0(t) = s_0(t) + n(t)$

Error if $y_o(t) > A/2$ $\qquad P_{e0} = \int_{A/2}^{\infty} \mathbf{N}(0, \sigma^2)\, dn$

Baseband system – "1" transmitted



Transmitted signal
$s_1(t)$

Noise signal
$n(t)$

Received signal
$y_1(t) = s_1(t) + n(t)$

Error if $y_1(t) < A/2$ $\qquad P_{e1} = \int_{-\infty}^{A/2} \mathbf{N}(A, \sigma^2)\, dn$

Baseband system – errors

- Possible errors:
  1. Symbol "0" transmitted, receiver decides "1"
  2. Symbol "1" transmitted, receiver decides "0"

- Total probability of error:

$$P_e = p_o P_{e0} + p_1 P_{e1}$$

Probability of "0" being sent

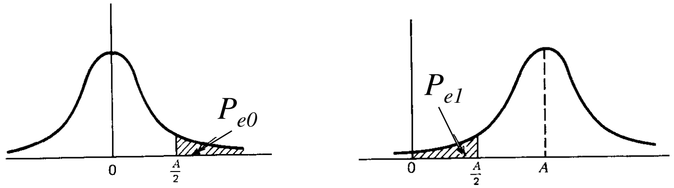
Probability of making an error if "0" was sent

Baseband system – errors

- For equally-probable symbols:

$$P_e = \frac{1}{2} P_{e0} + \frac{1}{2} P_{e1}$$

- Can show that $P_{e0} = P_{e1}$



- Hence, $P_e = ½\, P_{e0} + ½\, P_{e0} = P_{e0}$

$$P_e = \int_{A/2}^{\infty} \mathbf{N}(0, \sigma^2)\, dn$$

How to calculate P_e?

$$P_e = \int_{A/2}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{n^2}{2\sigma^2}\right) dn$$

1. Complementary error function (`erfc` in Matlab)

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-n^2)\, dn$$

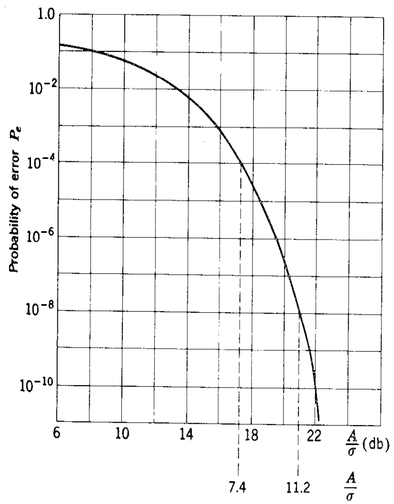
$$P_e = \tfrac{1}{2} \operatorname{erfc}\left(\frac{A}{\sigma\, 2\sqrt{2}}\right)$$

2. Q-function (tail function)

$$Q(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} \exp\left(\frac{-n^2}{2}\right) dn$$

$$P_e = Q\left(\frac{A}{2\sigma}\right)$$

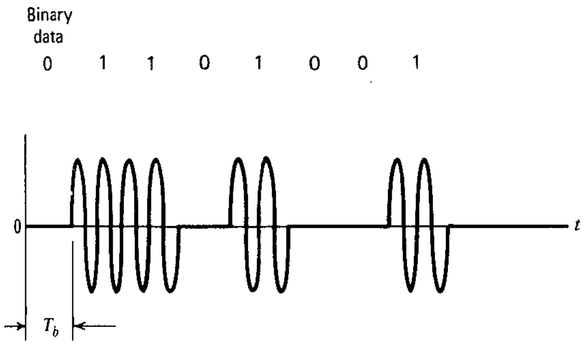
Baseband error probability



Example

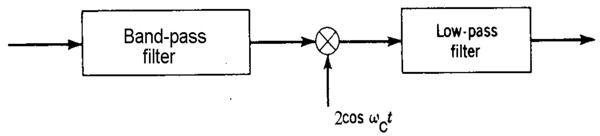
- Consider a digital system which uses a voltage level of 0 volts to represent a “0”, and a level of 0.22 volts to represent a “1”. The digital waveform has a bandwidth of 15 kHz.
- If this digital waveform is to be transmitted over a baseband channel having additive noise with flat power spectral density of $N_o/2 = 3 \times 10^{-8}$ W/Hz, what is the probability of error at the receiver output?

Amplitude-shift keying



$$s_0(t) = 0$$
$$s_1(t) = A\cos(\omega_c t)$$

Synchronous detector



- Identical to analog synchronous detector

ASK – "0" transmitted

- Predetection signal:

$$x_0(t) = \underbrace{0}_{\text{ASK "0"}} + \underbrace{n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)}_{\text{Bandpass noise}}$$

- After multiplier:

$$\begin{aligned} r_0(t) &= x_0(t) \times 2\cos(\omega_c t) \\ &= n_c(t) 2\cos^2(\omega_c t) - n_s(t) 2\sin(\omega_c t)\cos(\omega_c t) \\ &= n_c(t)[1 + \cancel{\cos(2\omega_c t)}] - \cancel{n_s(t)\sin(2\omega_c t)} \end{aligned}$$

- Receiver output:

$$y_0(t) = n_c(t)$$

ASK – "1" transmitted

- Predetection signal:

$$x_1(t) = \underbrace{A\cos(\omega_c t)}_{\text{ASK "1"}} + \underbrace{n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)}_{\text{Bandpass noise}}$$

- After multiplier:

$$\begin{aligned} r_1(t) &= x_1(t) \times 2\cos(\omega_c t) \\ &= [A + n_c(t)]\, 2\cos^2(\omega_c t) - n_s(t) 2\sin(\omega_c t)\cos(\omega_c t) \\ &= [A + n_c(t)][1 + \cancel{\cos(2\omega_c t)}] - \cancel{n_s(t)\sin(2\omega_c t)} \end{aligned}$$

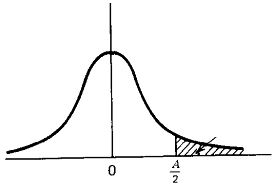
- Receiver output:

$$y_1(t) = A + n_c(t)$$

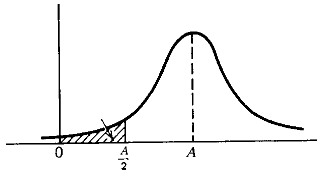
PDFs at receiver output

- ASK – “0” transmitted:
- ASK – “1” transmitted:

PDF of $y_0(t) = n_c(t)$



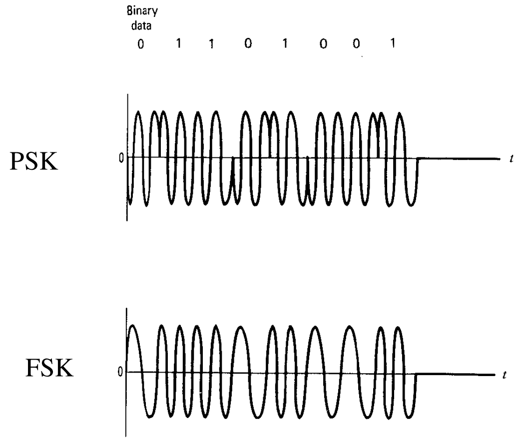
PDF of $y_1(t) = A + n_c(t)$



- Same as baseband!

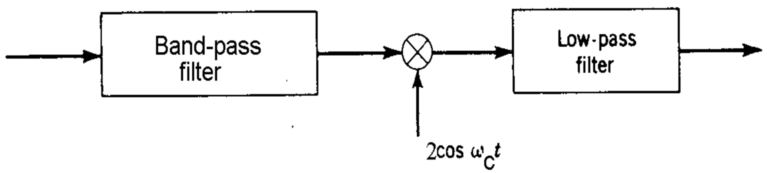
$$P_{e,\mathrm{ASK}} = \tfrac{1}{2}\operatorname{erfc}\left(\frac{A}{\sigma\, 2\sqrt{2}}\right)$$

Phase-shift keying



$$s_0(t) = -A\cos(\omega_c t)$$
$$s_1(t) = A\cos(\omega_c t)$$

PSK demodulator



- Band-pass filter bandwidth matched to modulated signal bandwidth
- Carrier frequency is ω_c
- Low-pass filter leaves only baseband signals

PSK – “0” transmitted

- Predetection signal:

$$x_0(t) = \underbrace{-A\cos(\omega_c t)}_{\text{PSK “0”}} + \underbrace{n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)}_{\text{Bandpass noise}}$$

- After multiplier:

$$\begin{aligned} r_0(t) &= x_0(t)\times 2\cos(\omega_c t) \\ &= \left[-A + n_c(t)\right] 2\cos^2(\omega_c t) - n_s(t) 2\sin(\omega_c t)\cos(\omega_c t) \\ &= \left[-A + n_c(t)\right]\left[1 + \cancel{\cos(2\omega_c t)}\right] - \cancel{n_s(t)\sin(2\omega_c t)} \end{aligned}$$

- Receiver output:

$$y_0(t) = -A + n_c(t)$$

PSK – "1" transmitted

- Predetection signal:

$$x_1(t) = \underbrace{A\cos(\omega_c t)}_{\text{PSK "1"}} + \underbrace{n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)}_{\text{Bandpass noise}}$$

- After multiplier:

$$\begin{aligned} r_1(t) &= x_1(t)\times 2\cos(\omega_c t) \\ &= [A + n_c(t)]\,2\cos^2(\omega_c t) - n_s(t)2\sin(\omega_c t)\cos(\omega_c t) \\ &= [A + n_c(t)][1 + \cancel{\cos(2\omega_c t)}] - \cancel{n_s(t)\sin(2\omega_c t)} \end{aligned}$$

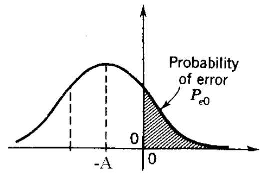
- Receiver output:

$$y_1(t) = A + n_c(t)$$

PSK – PDFs at receiver output

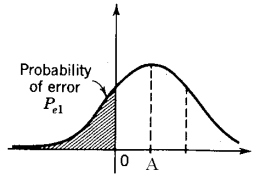
- PSK – "0" transmitted:

PDF of $y_0(t) = -A + n_c(t)$



- PSK – "1" transmitted:

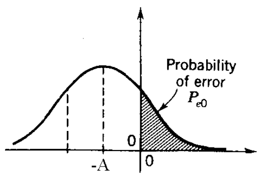
PDF of $y_1(t) = A + n_c(t)$



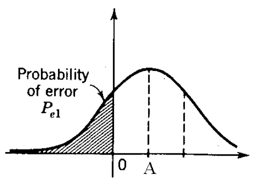
- Set threshold at 0:

$$\begin{aligned} &\text{if } \; y < 0, \text{ decide "0"} \\ &\text{if } \; y > 0, \text{ decide "1"} \end{aligned}$$

PSK – probability of error



$$\begin{aligned} P_{e0} &= \int_0^{\infty} \mathbf{N}(-A,\sigma^2) \\ &= \int_0^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(n+A)^2}{2\sigma^2}\right) dn \end{aligned}$$



$$\begin{aligned} P_{e1} &= \int_{-\infty}^{0} \mathbf{N}(A,\sigma^2) \\ &= \int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(n-A)^2}{2\sigma^2}\right) dn \end{aligned}$$

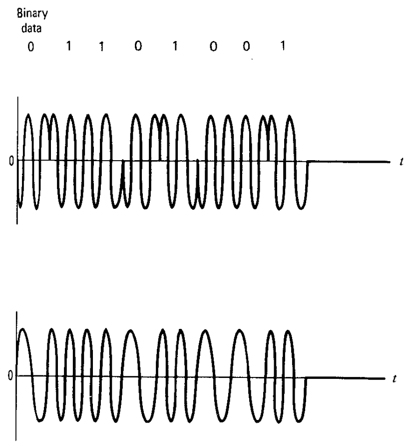
- Probability of bit error:

$$P_{e,\mathrm{PSK}} = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{\sigma\sqrt{2}}\right)$$

Frequency-shift keying

PSK

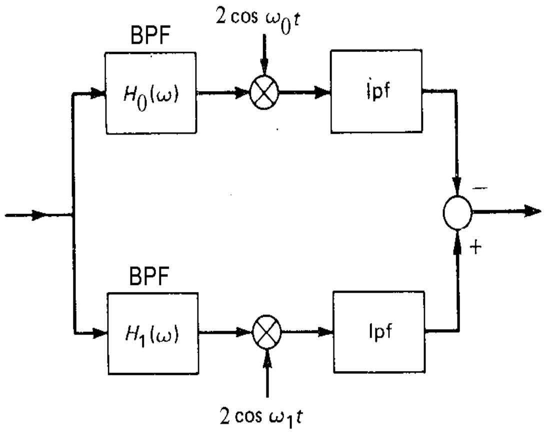
FSK



$$s_0(t) = A\cos(\omega_0 t)$$
$$s_1(t) = A\cos(\omega_1 t)$$

FSK detector

$$s_0(t) = A\cos(\omega_0 t)$$
$$s_1(t) = A\cos(\omega_1 t)$$



FSK

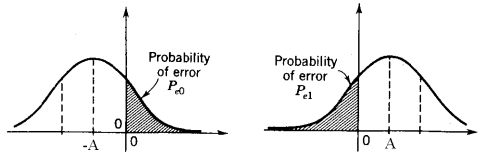
- Receiver output:

$$y_0(t) = -A + \left[n_c^1(t) - n_c^0(t)\right]$$

$$y_1(t) = A + \left[n_c^1(t) - n_c^0(t)\right]$$

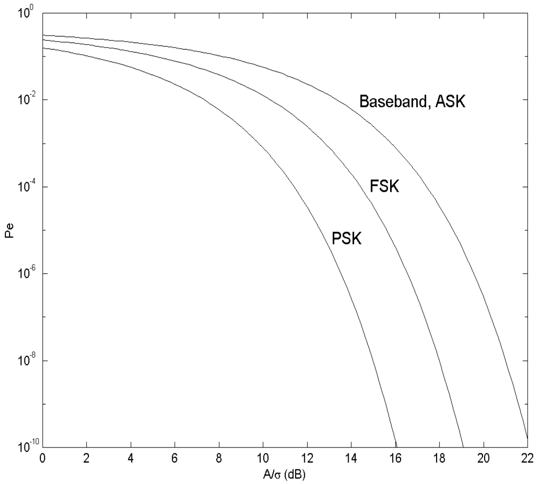
Independent noise sources, variances add

- PDFs same as for PSK, but variance is doubled:



$$P_{e,\text{FSK}} = \frac{1}{2}\text{erfc}\left(\frac{A}{2\sigma}\right)$$

Digital performance comparison



Summary

- For a baseband (or ASK) system:

$$P_e = \int_{A/2}^{\infty} \mathbf{N}(0,\sigma^2)\,dn = \tfrac{1}{2}\text{erfc}\left(\frac{A}{\sigma\, 2\sqrt{2}}\right)$$

- Probability of error for PSK and FSK:

$$P_{e,\text{PSK}} = \frac{1}{2}\text{erfc}\left(\frac{A}{\sigma\sqrt{2}}\right) \qquad P_{e,\text{FSK}} = \frac{1}{2}\text{erfc}\left(\frac{A}{2\sigma}\right)$$

- Comparison of digital systems:

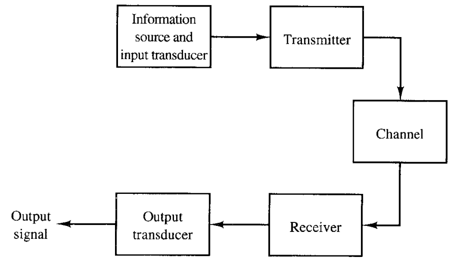
  PSK best, then FSK, ASK and baseband same

Lecture 8

- Information theory
  - Why?
  - Information
  - Entropy
  - Source coding (a little)

(See sections 5.1 to 5.4.2)

Why information theory?



- What is the performance of the "best" system?

"What would be the characteristics of an *ideal system*, [one that] is not limited by our engineering ingenuity and inventiveness but limited rather only by the fundamental nature of the physical universe"

Taub & Schilling

Information

- The purpose of a communication system is to **convey information from one point to another**
- What is information?
- Definition:

$$I(s) = \log_2\left(\frac{1}{p}\right) = -\log_2(p) \qquad \text{bits}$$

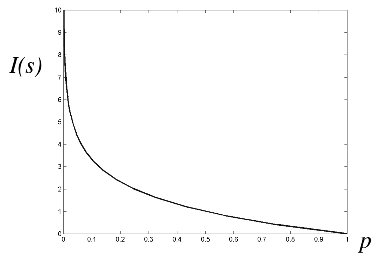
Information in symbol *s*

Probability of occurrence of symbol *s*

Conventional unit of information

Properties of *I(s)*

$$I(s) = -\log_2(p) \qquad \text{bits}$$



1. If $p=1$, $I(s)=0$
   (symbol that is certain to occur conveys no information)
2. $0<p<1,\ \infty<I(s)<0$
3. If $p=p_1\times p_2$, $I(s)=I(s_1)+I(s_2)$

Example

- Suppose we have two symbols:

$$s_0 = 0$$
$$s_1 = 1$$

- Each has probability of occurrence:

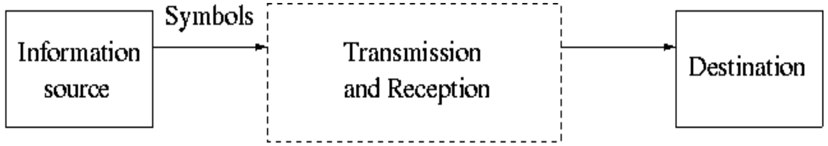
$$p_0 = p_1 = \tfrac{1}{2}$$

- Each symbol represents:

$$I(s) = -\log_2\left(\tfrac{1}{2}\right) = 1 \text{ bit of information}$$

- In this example, one symbol = one information bit, **but it is not always so!**

Sources and symbols



- Symbols:
  - may be binary (“0” and “1”)
  - can have more than 2 symbols, e.g. letters of the alphabet, etc.
- Sequence of symbols is random (otherwise no information is conveyed)
- Definition:

  If successive symbols are statistically independent, the information source is a **zero-memory source** (or **discrete memoryless source**)

- How much information is conveyed by symbols?

Entropy

- Definition:

$$H(S) = -\sum_{\text{all } k} p_k \log_2(p_k)$$

collection of all possible symbols

where $S = \{s_1, s_2, \ldots, s_K\}$ is **alphabet**

symbol

p_k is probability of occurence of symbol s_k

Note that: $\sum_{\text{all } k} p_k = 1 \leftarrow$ We’re certain that the symbol comes from the known alphabet

- Entropy: **average information per symbol**

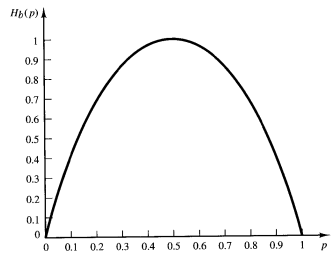
Example – binary source

- Alphabet:

$$S = \{s_0, s_1\}$$

- Probabilities: $p_0 = 1 - p_1$
- Entropy:

$$H(S) = -\sum_{\text{all } k} p_k \log_2(p_k)$$
$$= -(1 - p_1)\log_2(1 - p_1) - p_1 \log_2(p_1)$$



- How to represent (encode) each symbol?

  let $s_0 = 0, s_1 = 1$

  this requires 1 bit/symbol to transmit

Example – three symbol alphabet

- Alphabet:

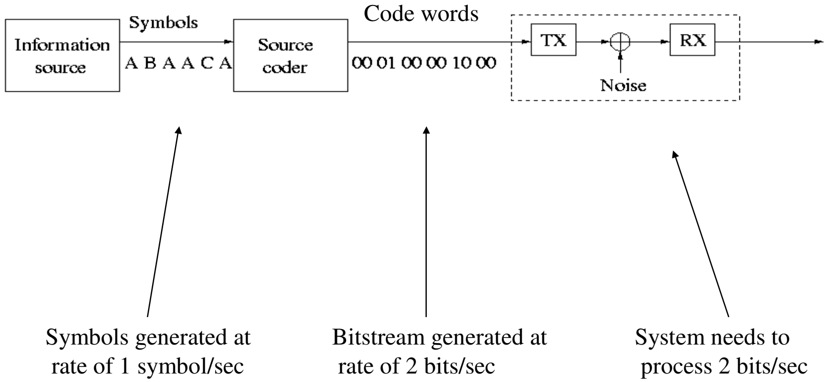
$$S = \{A, B, C\}$$

- Probabilities: $p_A = 0.7, \quad p_B = 0.2, \quad p_C = 0.1$
- Entropy: $$H(S) = -\sum_{\text{all } k} p_k \log_2(p_k)$$ $$= 1.157 \text{ bits/symbol}$$
- How to represent (encode) each symbol?

let $A = 00$
$B = 01$
$C = 10$

this requires 2 bits/symbol to transmit

Example – three symbol alphabet



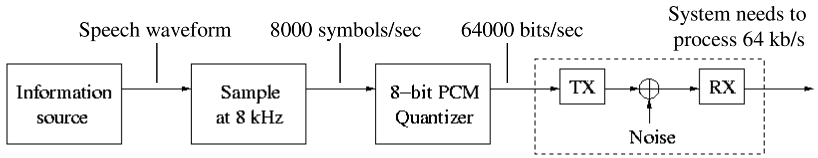
Source coding

- Amount of information we need to transmit, is determined (amongst other things) by **how many bits we need to transmit for each symbol**

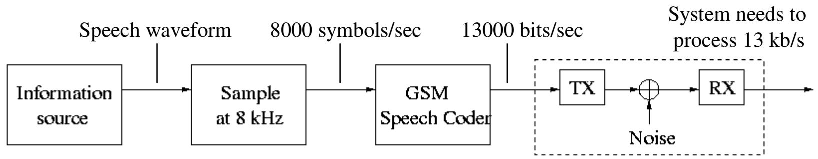
- In the binary case, only 1 bit required to transmit each symbol
- In the {A,B,C} case, 2 bits required to transmit each symbol

Examples

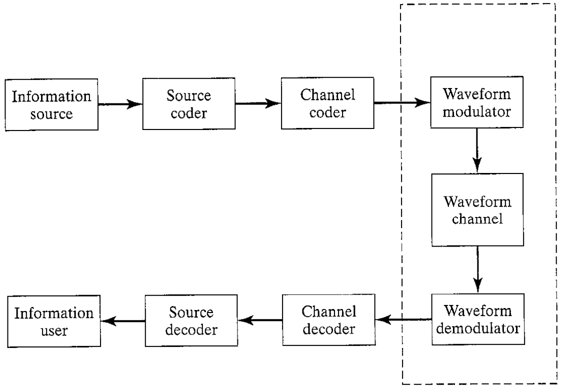
- Telephone:



- Cell phone:



Source vs channel coding



- Source coding: minimize the number of bits to be transmitted
- Channel coding: add extra bits to detect/correct errors

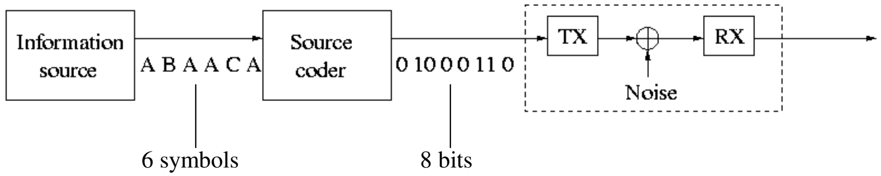
Source coding

- All **symbols do not need to be encoded with the same number of bits**

$$p_A = 0.7, \quad p_B = 0.2, \quad p_C = 0.1$$

- Example:

let $A = 0$
$B = 10$
$C = 11$



Average codeword length

- Definition:

$$\overline{L} = \sum_{\text{all } k} p_k \; l_k$$

Probability of occurrence of symbol s_k

Number of bits used to represent symbol s_k

- Example: $p_A = 0.7, \quad p_B = 0.2, \quad p_C = 0.1$

let $A = 0, \quad B = 10, \quad C = 11$

$$\begin{aligned}\overline{L} &= 0.7 \times 1 + 0.2 \times 2 + 0.1 \times 2 \\ &= 1.3 \quad \text{bits/symbol}\end{aligned}$$

Source coding

- Use variable-length code words
- Symbol that occurs frequently (i.e., relatively high p_k) should have short code word
- Symbol that occurs rarely should have long code word

Summary

- Information content (of a particular symbol):

$$I(s) = \log_2\left(\frac{1}{p}\right) = -\log_2(p) \qquad \text{bits}$$

- Entropy (for a complete alphabet, is the average information content per symbol):

$$H(S) = -\sum_{\text{all } k} p_k \log_2(p_k) \qquad \text{bits/symbol}$$

- Source coding:

  How many bits do we need to represent each symbol?

Lecture 9

- Source coding theorem
- Huffman coding algorithm

(See sections 5.4.2, 5.4.3)

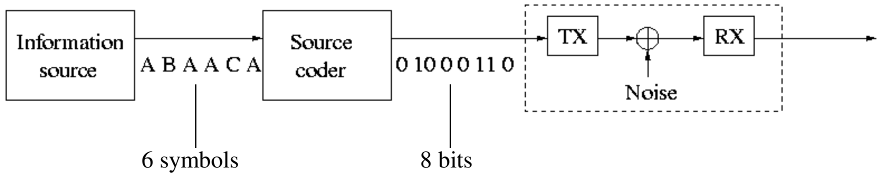
Source coding

- All symbols do not need to be encoded with the same number of bits
- Example:

$$p_A = 0.7, \quad p_B = 0.2, \quad p_C = 0.1 \longleftarrow \text{probabilities}$$

$$A = 0, \quad B = 10, \quad C = 11 \longleftarrow \text{code words}$$

$$\begin{aligned} \overline{L} &= 0.7\times 1 + 0.2\times 2 + 0.1\times 2 \\ &= 1.3 \quad \text{bits/symbol} \longleftarrow \text{average codeword length} \end{aligned}$$



Source coding

- How can we reduce the number of bits we need to transmit?
- What is the minimum number of bits we need for a particular symbol?
  (Source coding theorem)
- How can we encode symbols to achieve this minimum number of bits?
  (Huffman coding algorithm)

Equal probability symbols

- Example:

Alphabet: $S = \{A, B\}$

Probabilities: $p_A = 0.5, \quad p_B = 0.5$

Code words: $A = 0, \quad B = 1$

Requires 1 bit for each symbol

- In general, for n **equally-likely** symbols:

Probability of occurrence of each symbol is $p=1/n$

Number of bits to represent each symbol is

$$l = \log_2\left(\frac{1}{p}\right) = \log_2(n)$$

Unequal probabilities?

Alphabet: $S = \{s_1, s_2, \ldots, s_K\}$

Probabilities: $p_1, p_2, \ldots, p_K$

Any random sequence of N symbols (large N):

s_1: $N \times p_1$ occurrences

s_2: $N \times p_2$ occurrences

Particular sequence of N symbols:

$$S_N = \{s_1, s_2, s_1, s_3, s_3, s_2, s_1, \ldots\}$$

Probability of this particular sequence occurring:

$$p(S_N) = p_1 \times p_2 \times p_1 \times p_3 \times p_3 \times p_2 \times p_1 \times \ldots$$
$$= p_1^{Np_1} \times p_2^{Np_2} \times \ldots$$

Unequal probabilities ? (cont.)

Probability of any sequence of N symbols occurring:

$$p(S_N) = p_1^{Np_1} \times p_2^{Np_2} \times \ldots$$

Number of bits required to represent a sequence of N symbols:

$$l_N = \log_2\left(\frac{1}{p(S_N)}\right) = -\log_2\left(p_1^{Np_1} \times p_2^{Np_2} \times \ldots\right)$$
$$= -Np_1 \log_2(p_1) - Np_2 \log_2(p_2) - \ldots$$
$$= -N \sum_{\text{all } k} p_k \log_2(p_k) = N\ H(S)$$

Average number of bits for one symbol is:

$$\overline{L} = \frac{l_N}{N} = H(S)$$

Minimum codeword length

Source Coding Theorem:

For a general alphabet S, the minimum average codeword length is given by the entropy, *H(S)*.

- Significance:

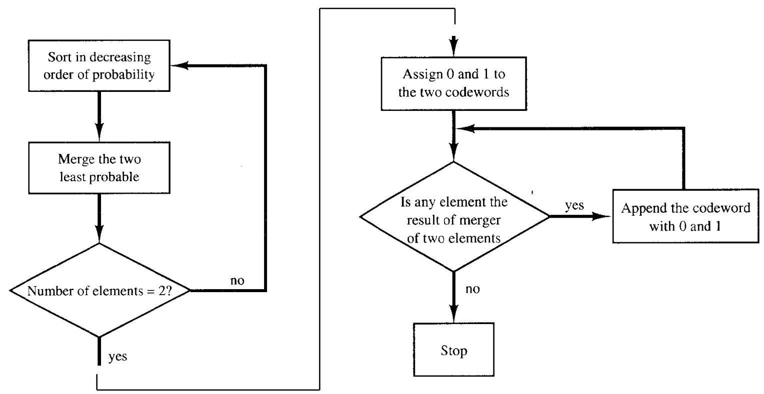
  For any practical source coding scheme, the average codeword length will always be greater than or equal to the source entropy, i.e.,

  $\overline{L} \geq H(S)$ bits/symbol

- How can we design an efficient coding scheme?

Huffman coding algorithm

- Optimum coding scheme – yields the shortest average codeword length



Example

- Consider a five-symbol alphabet having the probabilities indicated:

$$\text{Symbols}: A, B, C, D, E$$
$$\text{Probabilities}: p_A = 0.05, p_B = 0.15, p_C = 0.4, p_D = 0.3, p_E = 0.1$$

1. Calculate the entropy of the alphabet.
2. Using the Huffman algorithm, design a source coding scheme for this alphabet, and comment on the average codeword length achieved.

Huffman coding algorithm

- Uniquely decodable

  i.e., only one way to break bit stream into valid code words

- Instantaneous

  i.e., know immediately when a code word has ended

Summary

- Source coding theorem:

  For a general alphabet *S*, the minimum average codeword length is given by the entropy, *H(S)*.

- Huffman coding algorithm:

  Practical coding scheme that yields the shortest average codeword length

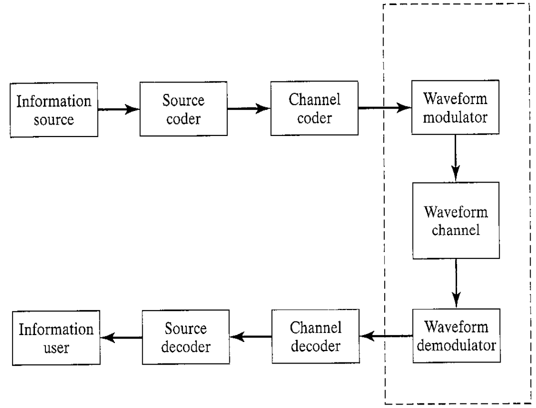
Lecture 10

- How much information can be reliably transferred over a noisy channel?
  (Channel capacity)

- What does information theory have to say about analog communication systems?

(See sections 5.5, 5.6)

Reliable transfer of information



- If channel is noisy, can information be transferred reliably?
- How much information?

Information rate

- Definition:

$$R = r\,H \quad \text{bits/sec}$$

R: Avg no. of information bits transferred per second
r: Avg. no. of symbols per second
H: Avg. no. of information bits per symbol

- Intuition:
  - R can be increased arbitrarily by increasing symbol rate r
  - For noisy channel, errors are bound to occur
  - Is there a value of R where probability of error is arbitrarily small?

Channel capacity

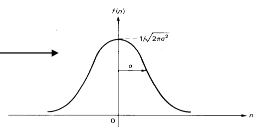
- Definition:
  Channel capacity, C, is maximum rate of information transfer over a noisy channel with arbitrarily small probability of error

Channel Capacity Theorem

If $R \le C$, then there exists a coding scheme such that symbols can be transmitted over a noisy channel with an arbitrarily small probability of error

Channel capacity

- Channel capacity theorem is a surprising result:
  - Gaussian noise has PDF



  - This is non-zero for all noise amplitudes
  - Sometimes (however infrequent) noise must over-ride signal → bit error
  - But, theorem says we can transfer information without error!!

- **Basic limitation due to noise is on speed of communication, not on reliability**

- So what is the channel capacity *C* ??

Channel capacity

Hartley-Shannon Theorem

For an additive white Gaussian noise channel, the channel capacity is:

$$C = B\log_2\left(1+\frac{P_S}{P_N}\right)$$

Average signal power at the receiver

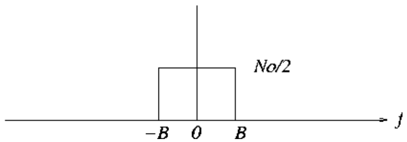
Bandwidth of the channel

Average noise power at the receiver

Example

- Consider a baseband system
- Noise power is:

$$P_N = \int_{-B}^{B}\frac{N_o}{2}\,df = N_oB$$



- Channel capacity:

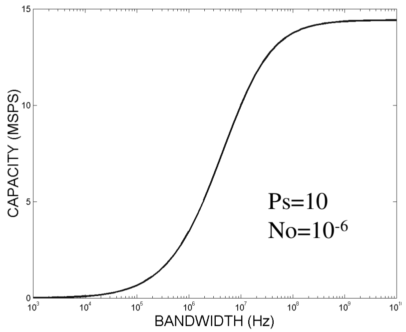
$$C = B\log_2\left(1+\frac{P_S}{N_oB}\right)$$

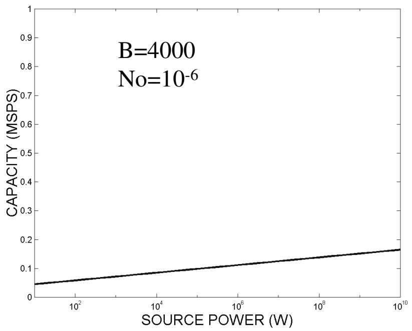
Example

- Consider a baseband channel with a bandwidth of *B*=4 kHz. Assume a message signal with an average power of *Ps*=10 W, is transmitted over this channel which has additive noise with a flat spectral density of height *No*/2 with *No*=10^{-6} W/Hz.

1. Calculate the channel capacity of this channel.
2. If the message signal is amplified by a factor of *n* before transmission, calculate the channel capacity when (a) *n*=2, and (b) *n*=10.
3. If the bandwidth of the channel is doubled to 8 kHz, what is the channel capacity now?

Example





Comments

$$C = B\log_2\left(1+\frac{P_S}{P_N}\right)$$

- More signal power increases capacity, but increase is slow
  Can increase capacity arbitrarily through P_S
- More bandwidth allows more symbols per second, **but** also increases the noise
- Can show that: $\lim_{B\to\infty} C = 1.44\frac{P_S}{N_o}$

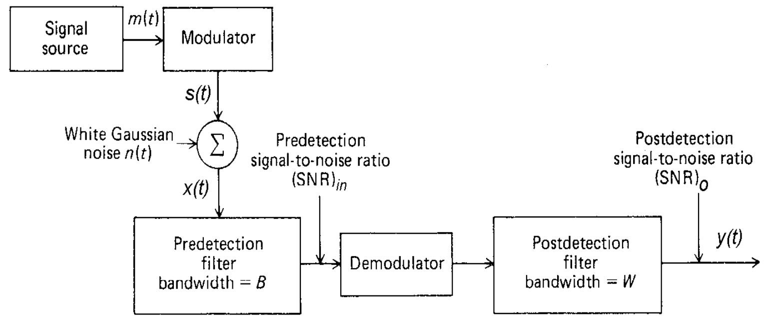
  Cannot increase capacity arbitrarily through B

More comments

$$C = B\log_2\left(1+\frac{P_S}{P_N}\right)$$

- This is capacity of an ideal “best” system
- How can we design something that comes close?
  - Through channel coding, modulation/demodulation schemes
  - But, no deterministic method exists to do it !

Information theory and analog



- Optimum communication system achieves the largest SNR at the receiver output

Optimum analog system

- Maximum rate that information can arrive at receiver:

  $$C_{in} = B\log_2\left(1 + SNR_{in}\right)$$

- Maximum rate that information can leave receiver:

  $$C_{out} = W\log_2\left(1 + SNR_{out}\right)$$

- Ideally, no information is lost:

  $$C_{out} = C_{in}$$

- Equating gives:

  $$SNR_{out} = \left(1 + SNR_{in}\right)^{B/W} - 1$$

- For any increase in bandwidth, output SNR increases exponentially

Optimum analog system

- Assume that channel noise is AWGN, having PSD: $N_o/2$
- Average noise power at demodulator input is: $P_N = N_o B$
- SNR at receiver input:

Transmitted power

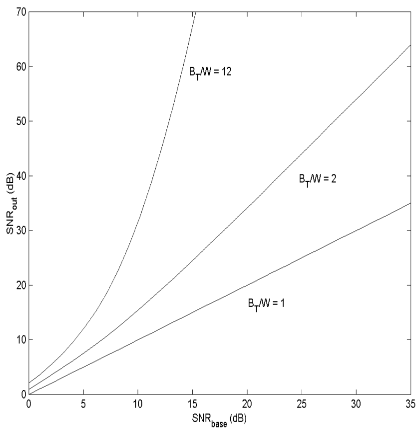
$$SNR_{in} = \frac{P_T}{N_o B} = \frac{W}{B}\overbrace{\frac{P_T}{N_o W}}^{\text{Baseband SNR}}$$

- SNR at receiver output:

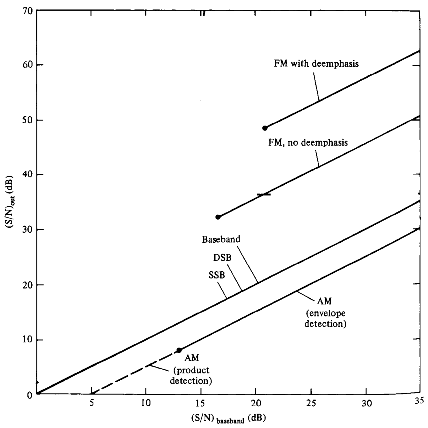
Bandwidth spreading ratio
transmission bw/message bw

$$SNR_{out} = \left(1 + \frac{W}{B}SNR_{base}\right)^{B/W} - 1$$

Analog performance



Ideal performance



Actual performance

Summary

- Information rate: $R = r\,H$ bits/sec
- Channel Capacity Theorem:

  If $R \le C$, then there exists a coding scheme such that symbols can be transmitted over a noisy channel with an arbitrarily small probability of error

- Hartley-Shannon Theorem (Gaussian noise channel):

  $$C = B\log_2\left(1 + \frac{P_S}{P_N}\right) \quad \text{bits/sec}$$

- Analog communication systems:

  Information theory tells us the best SNR