Problem Sheet 1 – Solutions

1. (a) From the definition

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

Let $T = nT_o$ giving

$$P = \lim_{n\to\infty} \frac{1}{nT_o} \int_{-nT_o/2}^{nT_o/2} |x(t)|^2 \, dt$$

But since the signal is periodic, the integral from $-nT_o/2$ to $nT_o/2$ is just n times the integral from $-T_o/2$ to $T_o/2$, i.e.,

$$\begin{aligned} P &= \lim_{n\to\infty} \frac{n}{nT_o} \int_{-T_o/2}^{T_o/2} |x(t)|^2 \, dt \\ &= \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x(t)|^2 \, dt \end{aligned}$$

(b) Substituting $x(t) = A\cos(2\pi f t + \theta)$, which has a period $T_0 = 1/f$, gives

$$\begin{aligned} P &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2 \cos^2(2\pi f t + \theta) \, dt \\ &= \frac{A^2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{1}{2} + \frac{1}{2}\cos(4\pi f t + 2\theta) \, dt \quad (*) \\ &= \frac{A^2}{T_0} \left[\frac{t}{2}\right]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \\ &= \frac{A^2}{2T_0} \left(\frac{T_0}{2} - \frac{-T_0}{2}\right) = \frac{A^2}{2} \end{aligned}$$

$(*)$ Integrating a cosine over any integer number of periods is zero.

2. (a) From the diagram, we see that $|x(t)|^2 = A^2$ for all t. The area under $|x(t)|^2$ over one period is A^2T, and after dividing by T we find that the average power is $P = A^2$.

(b) In this case

$$x(t) = At, \quad 0 < t < T$$

where T is the period. Thus,

$$\begin{aligned} P &= \frac{1}{T} \int_0^T A^2 t^2 \, dt \\ &= \frac{A^2}{T} \int_0^1 t^2 \, dt = \frac{A^2}{3} \left[t^3\right]_0^1 = \frac{A^2}{3} \end{aligned}$$

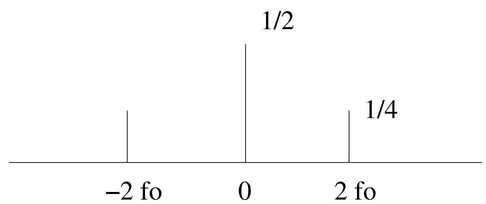
3. (a)

$$\begin{aligned} x(t) &= \cos^2(2\pi f_o t) = \frac{1}{2}\left[1 + \cos(4\pi f_o t)\right] \\ &= \frac{1}{2} + \frac{1}{4} e^{j4\pi f_o t} + \frac{1}{4} e^{-j4\pi f_o t} \end{aligned}$$

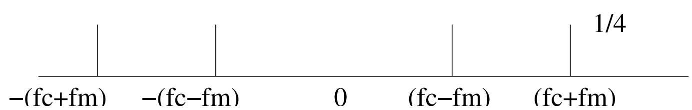
(b)

$$
\begin{aligned}
x(t) &= \frac{1}{2}\Big[\cos 2\pi(f_c+f_m)t + \cos 2\pi(f_c-f_m)t\Big] \\
&= \frac{1}{4}\Big[e^{j2\pi(f_c+f_m)t} + e^{-j2\pi(f_c+f_m)t} \\
&\qquad + e^{j2\pi(f_c-f_m)t} + e^{-j2\pi(f_c-f_m)t}\Big]
\end{aligned}
$$

(a)



(b)



Problem Sheet 2 – Solutions

1. The pdf of θ is

$$p(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \le \theta < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

By definition the mean is

$$\begin{aligned} E\{n(t)\} &= \int_{-\infty}^{\infty} A\cos(2\pi f_c t + \theta)\, p(\theta)\, d\theta \\ &= \frac{A}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t + \theta)\, d\theta \\ &= \frac{A}{2\pi} \times 0 = 0 \end{aligned}$$

By definition the mean square is

$$\begin{aligned} E\{n^2(t)\} &= \int_{-\infty}^{\infty} A^2 \cos^2(2\pi f_c t + \theta)\, p(\theta)\, d\theta \\ &= \frac{A^2}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2}\Big(1 + \cos(4\pi f_c t + 2\theta)\Big)\, d\theta \\ &= \frac{A^2}{4\pi} \left([\theta]_{-\pi}^{\pi} + \left[\frac{1}{2}\sin(4\pi f_c t + 2\theta)\right]_{-\pi}^{\pi} \right) \\ &= \frac{A^2}{2} \end{aligned}$$

2. For the upper branch

$$z_1 = 2n(t)\cos(\omega_c t)$$

But $n(t) = n_c(t)\cos(\omega_c t) - n_s(t)\sin(\omega_c t)$, so

$$\begin{aligned} z_1 &= 2n_c(t)\cos^2(\omega_c t) - 2n_s(t)\cos(\omega_c t)\sin(\omega_c t) \\ &= n_c(t)\big[1 + \cos(2\omega_c t)\big] - n_s(t)\sin(2\omega_c t) \end{aligned}$$

since $\cos^2 x = 1/2(1 + \cos 2x)$ and $\cos x \sin x = 1/2 \sin 2x$. The low-pass filter will remove all of the high-frequency terms at $2\omega_c$ leaving only the $n_c(t)$ term.

Similarly,

$$\begin{aligned} z_2 &= -2n(t)\sin(\omega_c t) \\ &= -2n_c(t)\cos(\omega_c t)\sin(\omega_c t) + 2n_s(t)\sin^2(\omega_c t) \\ &= -n_c(t)\sin(2\omega_c t) + n_s(t)\big[1 - \cos(2\omega_c t)\big] \end{aligned}$$

since $\sin^2 x = 1/2(1 - \cos 2x)$. Again, low-pass filtering will remove the $2\omega_c$ terms leaving only $n_s(t)$.

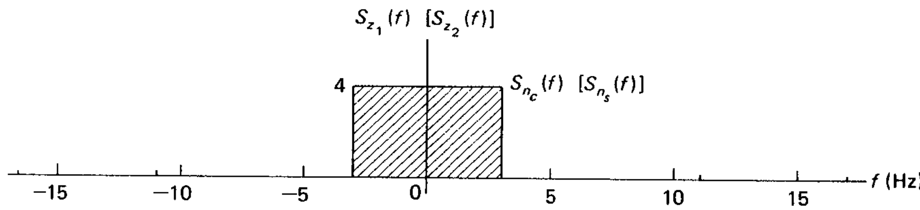
3. Recall that

$$n_c(t) = \sum_k a_k \cos(2\pi(f_k - f_c)t + \theta_k)$$

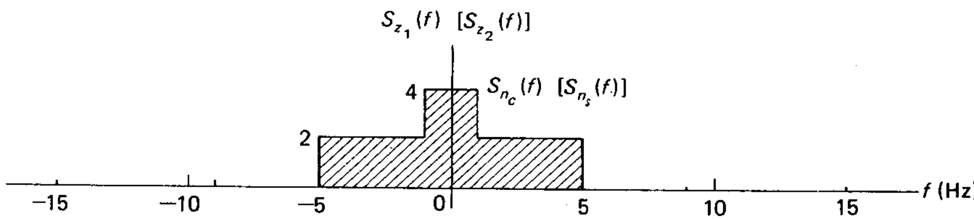
Hence, it consists of terms at frequencies $f_k - f_c$, where f_k are the frequencies present in the bandpass noise signal.

In the following figures, the shaded region is the PSD of $n_c(t)$.

(a)



(b)



Problem Sheet 3 – Solutions

1. In this question it is very important to note that the channel noise is not white Gaussian noise. Therefore we can not directly use the expressions derived in Chapter 3 of the notes (since these are only valid for white noise). Also note that we are considering the *predetection* SNR at the receiver (not the output SNR as in the notes).

   If the channel attenuates the transmitted signal by 40 dB, then the received signal power will be 10^{-4} times the transmitted signal power.

   (a) For baseband, the transmitted signal power is $P_T = 10$ W (i.e., the same as the message power). The received signal power is therefore $P_R = 1$ mW.

   The noise power is found by integrating the noise PSD over the transmission bandwidth:

$$P_N = 2\int_0^{10^4} \mathbb{S}(f)\,df = 2N_o \int_0^{10^4} 1 - \frac{f}{200\times 10^3}\,df$$
$$= 2N_o \times 10^4 \times 0.975 = 19.5\ \mu W$$

   This gives an SNR at the receiver input of

$$\text{SNR} = \frac{1\times 10^{-3}}{19.5\times 10^{-6}} = 17.1 \text{ dB}$$

   (b) For DSB-SC, the transmitted signal power (3.15) is $P_T = A_c^2 P/2 = 5$ W. The received signal power is therefore $P_R = 0.5$ mW.

   The noise power is:

$$P_N = 2\int_{f_c-10^4}^{f_c+10^4} \mathbb{S}(f)\,df = 2N_o \times 0.5 \times 20 \times 10^3$$
$$= 20\ \mu W$$

   giving a receiver input SNR of

$$\text{SNR} = \frac{0.5\times 10^{-3}}{20\times 10^{-6}} = 14 \text{ dB}$$

   (c) With a different carrier frequency, the noise power becomes:

$$P_N = 2N_o \times 0.25 \times 20 \times 10^3 = 10\ \mu W$$

   and the receiver input SNR is

$$\text{SNR} = \frac{0.5\times 10^{-3}}{10\times 10^{-6}} = 17 \text{ dB}$$

   Observe that these results are quite different from what one would obtain if the channel had white Gaussian noise. If the noise PSD was flat, then the SNR of DSB-SC would be independent of the carrier frequency.

2. (a) After the BPF, the signal is

$$x(t) = s(t) + n(t)$$

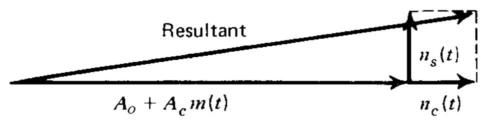
where $s(t) = A_c m(t) \cos \omega_c t$ and $n(t)$ is bandpass noise. Thus

$$x(t) = [A_c m(t) + n_c(t)] \cos \omega_c t - n_s(t) \sin \omega_c t$$

After the summation, the signal is

$$y(t) = [A_c m(t) + n_c(t) + A_o] \cos \omega_c t - n_s(t) \sin \omega_c t$$

The phasor diagram of this signal is shown below.



This is the same phasor diagram as for the envelope detector in Section 3.3.4. The output of the envelope detector (for A_o large) is $\tilde{y}(t) \approx A_o + A_c m(t) + n_c(t)$. After the blocking capacitor the final output is $A_c m(t) + n_c(t)$ which is identical to the output of the product detector in Section 3.3.2.

(b) The signal power at the receiver output is

$$P_S = A_c^2 P$$

where P is the average power in the message signal. The noise power at the receiver output is

$$P_N = 2\frac{N_o}{2} B_T$$

So the output SNR is

$$\text{SNR}_o = \frac{A_c^2 P}{N_o B_T}$$

3. In Section 3.3.4 of the course notes, the output SNR of an envelope detector (for small noise) was shown to be

$$\text{SNR} = \frac{P}{A^2 + P}\,\text{SNR}_{\text{baseband}}$$

The modulation index is defined as $\mu = m_p/A$ where m_p is the peak value of the message signal. Substitution gives

$$\text{SNR} = \frac{P}{m_p^2/\mu^2 + P}\,\text{SNR}_{\text{baseband}}$$

Now, increasing the value of μ will decrease the term in the denominator, thereby increasing the SNR. But, μ cannot be increased arbitrarily, since for an envelope detector to operate the modulation index $\mu \leq 1$. Thus, the value of μ that gives the maximum SNR is $\mu = 1$.

The resulting SNR expression is

$$\text{SNR} = \frac{P}{m_p^2 + P}\,\text{SNR}_{\text{baseband}}$$

Problem Sheet 4 – Solutions

1. (a) From Section 4.3.2, the error probability for ASK is

$$P_e = \frac{1}{2}\,\text{erfc}\left(\frac{A}{\sigma 2\sqrt{2}}\right)$$

where $A = 0.7$ is carrier amplitude and $\sigma = 0.125$ is noise standard deviation. substitution yields $P_e = 2.6 \times 10^{-3}$ using the MATLAB `erfc` function.

Alternatively it can be approximated from Figure 4.8 using $A/\sigma = 0.7/0.125 = 15$ dB.

(b) For PSK, Section 4.3.3 gives

$$P_e = \frac{1}{2}\,\text{erfc}\left(\frac{A}{\sigma\sqrt{2}}\right) = 1.07 \times 10^{-8}$$

Again, the probability of error can also be approximated from Figure 4.8.

2. (a) From Section 4.4.2, a linear quantizer has a mean square error of $P_N = \Delta^2/12$ where Δ is the separation between quantizer levels. We're told the input signal has a Gaussian pdf with std deviation σ, so the average power in the source signal is $P_S = \sigma^2$. The quantization step size is chosen such that the range covered by the 2^n quantizing levels is 8σ, giving a quantization step size of

$$\Delta = \frac{8\sigma}{2^n - 1} \approx \frac{8\sigma}{2^n}$$

Substitution yields

$$\begin{aligned}
P_N &= \frac{64\sigma^2}{12 \times 2^{2n}} = \frac{16}{3}\,\frac{\sigma^2}{2^{2n}} \\
\text{SNR} &= \frac{P_S}{P_N} = \frac{3}{16} \times 2^{2n} \\
10\log_{10}\text{SNR} &= 10\log_{10}\left(\frac{3}{16}\right) + 20n\log_{10} 2 \\
&= -7.3 + 6.02\,n
\end{aligned}$$

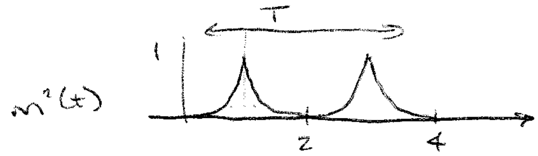
(b) For the probability of overload we want to find $p(x > 4\sigma) + p(x < -4\sigma)$ for a Gaussian r.v. with mean 0 and variance σ^2. Because the pdf is symmetric, this is

$$\begin{aligned}
P_{\text{overload}} &= 2\,p(x > 4\sigma) \\
&= 2\int_{4\sigma}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left(\frac{-x^2}{2\sigma^2}\right)\,dx \\
&= \text{erfc}\left(\frac{4\sigma}{\sigma\sqrt{2}}\right) = 6.3 \times 10^{-5}
\end{aligned}$$

Note that the last line follows from the change of variable $z = x/(\sigma\sqrt{2})$ (see equations (4.7) and (4.8) of the notes).

3. With $m^2(t)$ shown below, the signal power is:

$$
\begin{aligned}
P_S &= \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} m^2(t)\ dt = \frac{4}{T}\int_0^1\ m^2(t)\ dt \quad \text{(due to symmetry)} \\
&= \int_0^1\ t^2\ dt = \frac{1}{3}
\end{aligned}
$$



The noise power is:

$$
P_N = \frac{\Delta^2}{12} = \frac{(\frac{2}{2^n})^2}{12} = \frac{4\times 2^{-2n}}{12} = \frac{2^{-2n}}{3}
$$

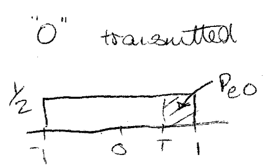
The signal to noise ratio is:

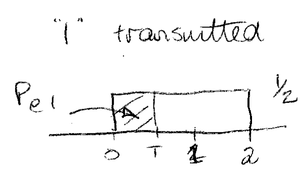
$$
\text{SNR} = \frac{1/3}{2^{-2n}/3} = 2^{2n}
$$

or

$$
\text{SNR}_{\text{dB}} = 10\log_{10}\left(2^{2n}\right) = 20n\log_{10}2 = 6.02n
$$

4. The pdf's of the received signals are as shown below, with $P_{e0} = 0.5(1-T)$ and $P_{e1} = 0.5T$.





For equally-likely symbols,

$$
\begin{aligned}
P_e &= p_0P_{e0} + p_1P_{e1} = \frac{1}{2}\left(P_{e0}+P_{e1}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(1-T)+\frac{1}{2}T\right) = \frac{1}{4}(1-T+T) = 0.25
\end{aligned}
$$

Problem Sheet 5 – Solutions

1. From Example 5.1 we have

$$H = -(1-p_1)\log_2(1-p_1) - p_1 \log_2 p_1$$

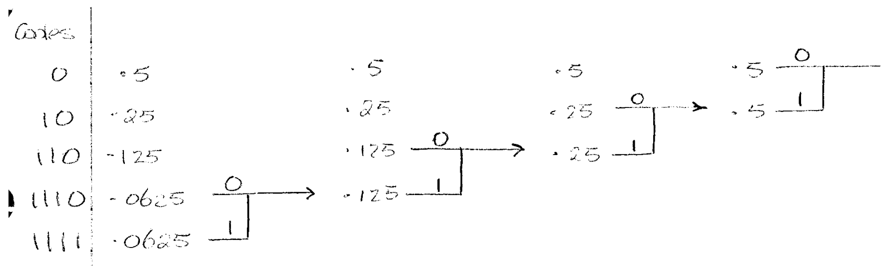
Without coding a binary bit stream would require 1 bit / symbol.

(a) For $p_1 = 0.5$ substitution gives $H = 1$ bit/symbol. Hence, no compression is possible.

(b) For $p_1 = 0.2$ substitution gives $H = 0.7219$ bits/symbol.

(c) For $p_1 = 0.1$ substitution gives $H = 0.08$ bits/symbol.

2. (a) The Huffman procedure is shown below.



The average codeword length is

$$\bar{L} = \sum_k p_k \ell_k$$

where p_k is the probability of occurrence of the kth symbol, and ℓ_k is the number of bits used to encode the kth symbol. Hence,

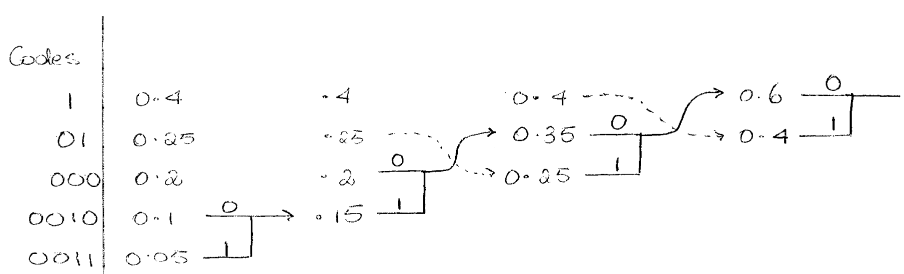
$$\begin{aligned}\bar{L} &= 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.0625 \times 4 + 0.0625 \times 4 \\ &= 1.875 \ \text{ bits/symbol}\end{aligned}$$

The source entropy is

$$\begin{aligned}H &= -\sum_k p_k \log_2 p_k \\ &= -.5\log_2 .5 - .25\log_2 .25 - .125\log_2 .125 - .0625\log_2 .0625 - .0625\log_2 .0625 \\ &= 1.875 \ \text{ bits/symbol}\end{aligned}$$

In this case $\bar{L} = H$, because of the exact equidivision of probabilities at every step.

(b) The Huffman procedure is shown below.



In this case the average codeword length is

$$\bar{L} = 2.1 \ \text{bits/symbol}$$

and the source entropy is

$$H = 2.0414 \ \text{bits/symbol}$$

In this case (and for most symbol probabilities), we have $\bar{L} > H$.

3. (a) Number of bits required, $n = \log_2 110 = 6.78 \approx 7$ bits.

   (b) The required SNR as a power ratio is 20 dB = 100. Substitution into the Hartley-Shannon equation gives

$$\begin{aligned} C &= B \log_2(1 + \text{SNR}) \\ &= 3200 \log_2 101 = 2.13 \times 10^4 \text{bits/sec} \\ &= 3.04 \times 10^3 \text{chars / sec} \end{aligned}$$

4. Each group of 2 symbols can be considered as a symbol in a new alphabet, which will have probabilities:

$$\begin{aligned} AA &: 3/4 \times 3/4 = 0.5625 \\ AB &: 3/4 \times 1/4 = 0.1875 \\ BA &: 1/4 \times 3/4 = 0.1875 \\ BB &: 1/4 \times 1/4 = 0.0625 \end{aligned}$$

The entropy of the new alphabet is $H = 1.6226$ bits per 2 symbols, or $H = 0.8113$ bits per symbol.

The average code length is

$$\begin{aligned} L &= \sum p_k \ell_k \\ &= .5625 \times 1 + .1875 \times 2 + .1875 \times 3 + .0625 \times 3 \\ &= 1.68745 \text{ bits per 2 symbols} = 0.84375 \text{ bits per symbol} \end{aligned}$$

Since $L > H$, this code is not optimum. The efficiency is $0.8113/0.84375 = 0.96$.