Let us revisit sine and cosine waves. A sine wave can be completely defined with three parameters $V_p$, the peak voltage (or amplitude), its frequency $w$ in radians/second or $f$ in cycles/second (Hz), and the phase angle $\phi$.

Cosine waves are the same as sine waves, except that cosine wave is 90 degrees or $\pi/2$ radians advance in phase. That is:

$$v(t) = V_p \sin(2\pi ft + \phi) = V_p \cos(2\pi ft + \phi + \frac{\pi}{2})$$

It is worth remembering that one cycle of a sine or cosine wave has a phase angle value of $2\pi$ radians or 360 degrees.
Resistors obey Ohm’s Law. The ratio of voltage to current through a resistor remains a constant no matter what frequency is the signal. This is not true with capacitors and inductors. As shown here the ratio $V/I$ when driven by a sine wave current source results in a voltage that is a cosine wave.

Consider the case of a capacitor. The capacitor voltage $V_C(t)$ is INVERSELY proportional to the frequency of the sinusoidal signal. If $\omega \to \infty$, $V_C(t) \to 0$.

Conversely, for inductor, $\omega \to \infty$, $V_L(t) \to \infty$.

Let us for the moment just consider ratio of the peak magnitudes of the voltage and the current in the three cases.

The ratio is simply $R$ for resistor.

The ratio is $1/\omega C$ for capacitor.

The ratio is $\omega L$ for inductor.
For capacitors and inductors, this ratio of peak voltage over peak current is frequency dependent. They are called reactance.

Both resistance and reactance are measures of how the components oppose the flow of current. The unit of reactance is the same as that of resistance – in ohms.

We use the symbol X to represent reactance here.

Reactance is the equivalent of "resistance" in a resistor, except two things: 1) reactance is for capacitor or inductor; 2) reactance magnitude changes with frequency of the electrical signal.

Furthermore, when we use reactance, we only consider peak voltage and current in a component because we only use the maximum (or peak) voltage and current in the calculation. Therefore, how can we use reactance in nodal analysis? The answer is to use complex number representation.

When you charge a capacitor with sinewave current, the voltage is a cosine wave at the same frequency (see slide 3). In other words, the current and the voltage through a capacitors are related: they differs from each other by 90 degrees or \( \pi / 2 \) radians.

\[
\sin(\omega t) = \cos(\omega t + \pi / 2)
\]

Let us now introduce a modified reactance to take this into account: the idea of impedance.

Impedance of a capacitor: \( \frac{v_c}{i_c} = \frac{1}{j\omega C} \)

Impedance of an inductor: \( \frac{v_i}{i_i} = j\omega L \)
Reactive components are things like capacitors and inductors. Their reactance (equivalent to resistance in resistors) are frequency dependent. This frequency dependency turns out to be very useful in building interesting circuits, e.g. filters which provide frequency selectivity.

Before we look at these circuits, I want to introduce to idea of two port network and gain.

A two-port network as an input port to which we apply stimulus $V_i$. There is an output port that provides a signal $V_O$. The ratio $V_O/V_i$ is the voltage gain.

Using complex number algebraic methods, we can easily work out the voltage gain of this simple RC network. Note that the gain equation is frequency dependent (i.e. it is a function of $\omega$).

The relationship between output $Y$ and input $X$ as a function of signal frequency $\omega$ is known as frequency response.

From the gain equation, we can compute the magnitude of the gain as a function of frequency. We can also plot the phase difference (output relative to input) as a function of frequency. The former is known as the magnitude (or amplitude) response. The latter is known as the phase response.

In the literature, graphs showing gain magnitude and phase vs frequency is also known as "Bode diagrams".

On this course, we will mainly focus on the magnitude response, and we will ignore the phase response most of the time.
Here is a circuit with $R = 10k$ and $C = 1 \text{ microfarad}$, and how the gain magnitude and phase changes with frequency. Note that at $\omega = 1/RC$, the gain is $0.71$ (or $1/\sqrt{2}$). This will come up again later.

Percentage differences are often more important than absolute differences. Therefore it is often more revealing if we plot frequency and gain in logarithmic scales.

For voltages, we normally calculate the gain in terms of decibels (dB) as defined here.

Remember that dB is dimensionless – it is a scaling, not a unit.

The definition of Gain in decibel is:

$$\text{decibels (dB)} = 20 \log_{10} \left| \frac{V_2}{V_1} \right|$$

When expressing power gain, the formula is different – the constant before the log is 10x not 20x.
I now want to show you how to make informed guess to the magnitude response of a circuit from its gain equation without having to do any calculations. This gain equation is frequency dependent and is often written as $H(j\omega)$. Since it is the ratio of output voltage to input voltage (or output current to input current), it is called a **TRANSFER FUNCTION**:

$$Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

Now consider the function: $H(j\omega) = a \cdot j\omega + b$.
If the magnitude $|a\omega|$ is much smaller than $|b|$, $|H(j\omega)| \approx |b|$.
If the magnitude $|a\omega|$ is much larger than $|b|$, $|H(j\omega)| \approx |a\omega|$.

Let us take the RC circuit as shown here. For low frequencies, the magnitude is 1 or 0 dB.

For high frequencies the magnitude drops linearly with $\omega$ (i.e. it is proportional to $\omega^{-1}$). So, in terms of dBs, over one decade (factor of 10), it falls by 20 dB. The slope of the line at high frequency is therefore -20 dB/decade.

How high must the frequency be before it is called "high"?
The intersection of these two line is at $\omega = 1/RC$. This is known as the corner frequency.

Here is another example of using the estimation method to plot the magnitude of the transfer function $|H(j\omega)|$. It does not matter where the function comes from, and it is complicated. Yet with our approximation method, you can plot the rough shape of the magnitude response.
Here is another example where we add a resistor $R$ in series with $C$ as shown.

Impedance of $R$ and $C$ in series is:

$$Z_{RC} = R + \frac{1}{j\omega C}$$

Using voltage divider equation:

$$\frac{Y}{X} = \frac{Z_{RC}}{3R + Z_{RC}} = \frac{R + \frac{1}{j\omega C}}{4R + \frac{1}{j\omega C}}$$

Multiply top and bottom by $j\omega C$:

$$\frac{Y}{X} = \frac{1 + j\omega RC}{1 + j\omega 4RC}$$

**Magnitude Response:**

- **Gradient Changes:** $-20$ dB/dec at $\omega = \frac{1}{4RC}$ and $+20$ at $\omega = \frac{1}{RC}$
- **Line equations:** $H(j\omega) = (a) 1$, \hspace{1cm} (b) $\frac{1}{4\omega RC}$, \hspace{1cm} (c) $\frac{j\omega RC}{4j\omega RC} = \frac{1}{4}$

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Let us take another look at the simple RC network. Remember that the impedance of a capacitor is inversely proportional to frequency. Therefore at low frequency, a capacitor appears as open-circuit. At high frequency, it appears as short-circuit.

Using the principle of voltage divider, this circuit will give you a low output at high frequency ($X_C$ is small), and does not attenuate the signal at low frequency ($X_C$ is large). We call this a low pass filter (LF).

The transfer function $H(j\omega) = \frac{Y}{X}$ has only one $j\omega$ term in the denominator. We call this first order filter.

The order of a filter is the highest power of $j\omega$ in the denominator of the transfer function (Gain function).

A first order filter will have a roll-off (i.e. rate of drop in gain) of $-20$ dB/decade.

An nth order filter will have a roll-off of $-20n$ dB/decade.
We have seen this before. Here we consider it as a filter. If $R = 0$, it is similar to the simple RC network we have seen in the last slide. However, we have added a $R$ in series with $C$. Introducing the resistor $R$ in series with $C$ adds an extra corner frequency at $q$. Now the gain drops off initially at -20dB/decade after the first corner frequency at $1/4RC$. However, at $1/RC$, the gain levels out as the impedance of $C$ is now negligible, and the circuit appears like a simple resistor voltage divider.

If you swap the RC and to form a CR circuit as shown, we have a magnitude response where at high frequency, $C$ appears to be short circuit and $Y=X$. However $C$ blocks any low frequency and DC signals. Therefore we now have a high pass filter.

Again you can work out the straight line approximation with the gain equation (transfer function).
This is a circuit including a R in series with L and C in parallel. I want you to try to derive the Y/X equation at home and prove that it is as given here. I will not go into details for this circuit.

Generally speaking, the order of the filter is the number of C + number of L. So this circuit is a 2\textsuperscript{nd} order filter. This is seen from the gain equation which has \((j\omega)^2\) term in the denominator.

The frequency at which the gain is maximum is called the resonant frequency. At this frequency, the impedance (and reactance) of L || C circuit is approaching infinity (open circuit). Therefore this network will pass through only signals at the resonant frequency and suppress signals of all other frequencies. It is called a band pass filter.