

Lecture 11

Discrete Time Signals

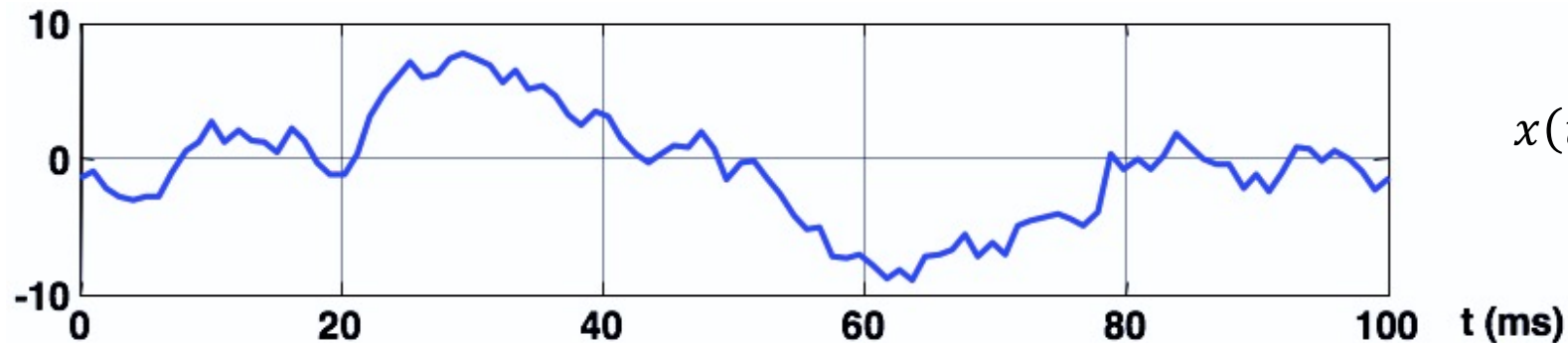
Prof Peter YK Cheung
Dyson School of Design Engineering



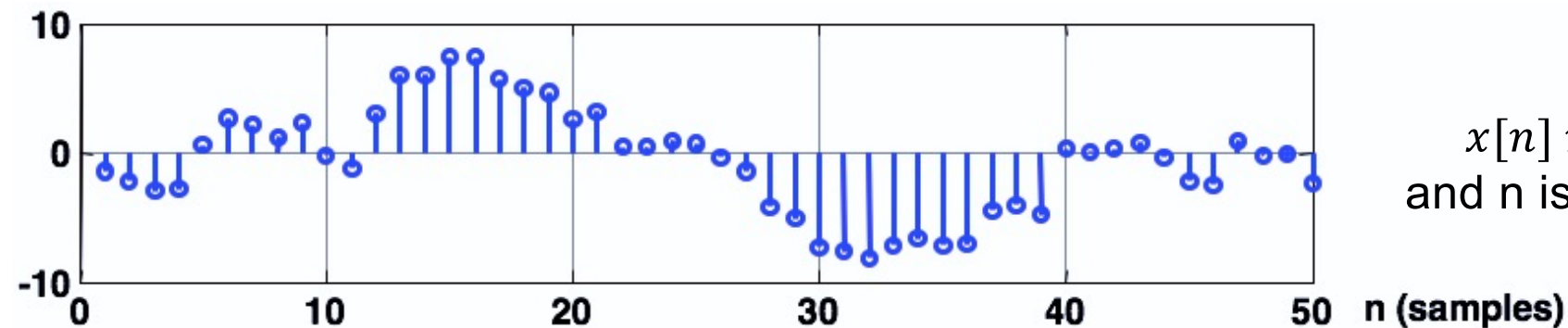
URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

Discrete time signals

- ◆ Sampling a continuous time signal at frequency f_s converts the signal to discrete time.
- ◆ How frequently we need to sample is governed by the Sampling Theorem.



$$x(t) \quad t \geq 0$$

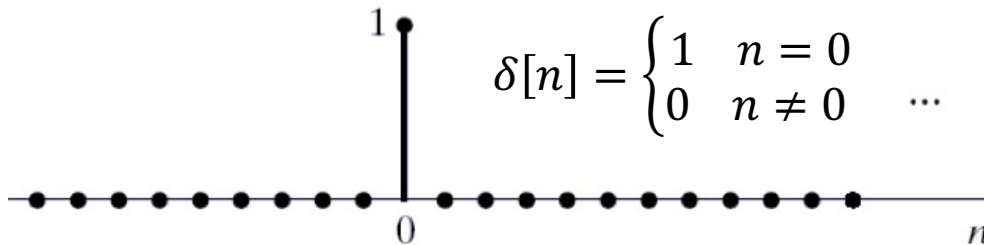


$$x[n] \quad n \geq 0$$

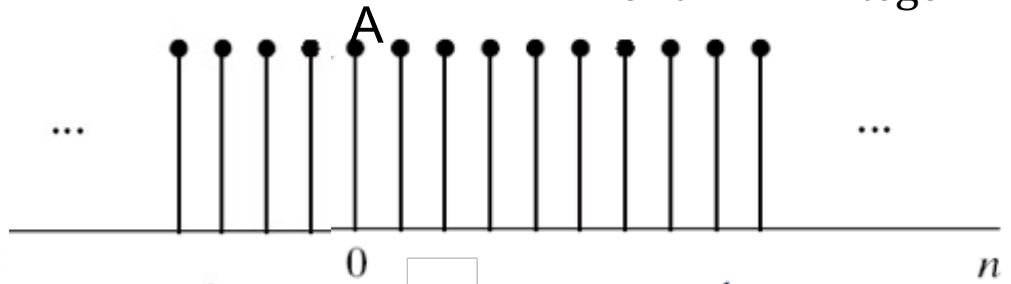
and n is integer

Basic discrete signals

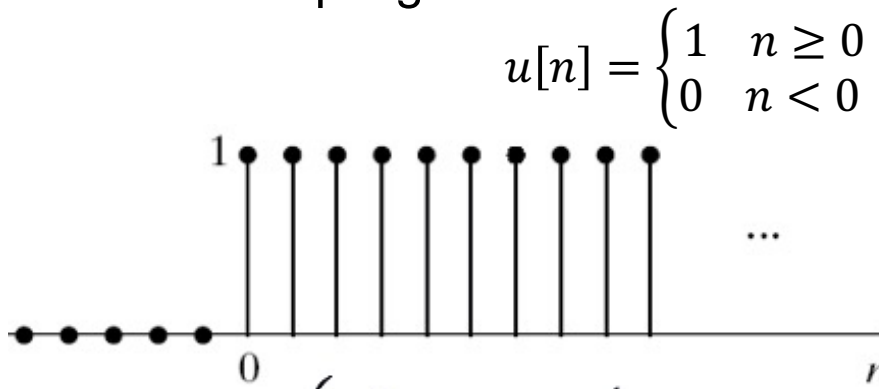
Impulse signal



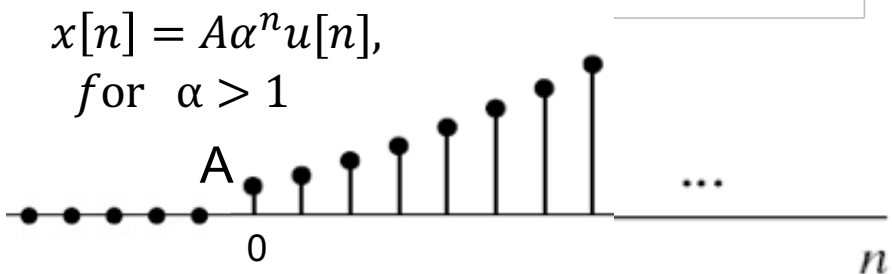
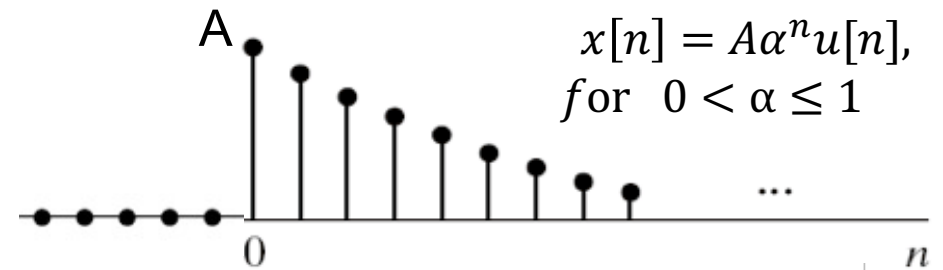
$$x[n] = A, \text{ for all } n = \text{integer}$$



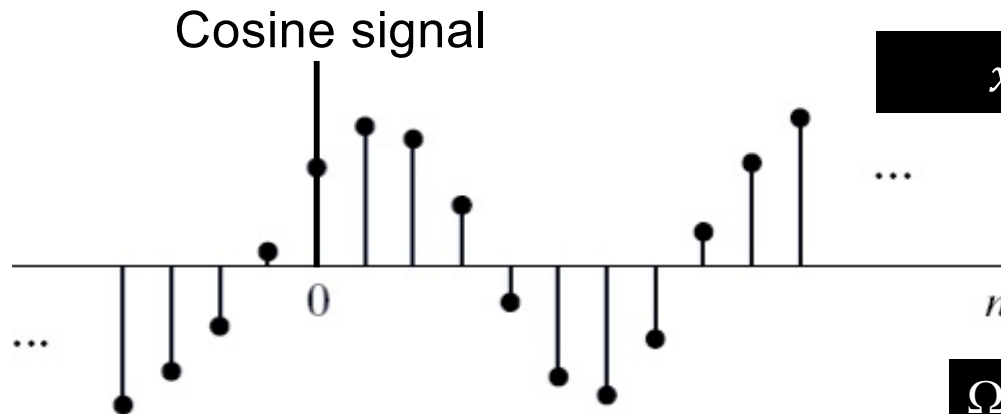
Unit step signal



Causal exponential signals



Discrete sinusoidal signal



$$x[n] = A \cos(\Omega_0 n + \phi) \quad \text{for } n = \text{integer}$$

Ω_0 is angle increment between samples
In radian/sample

- ◆ Compare this with continuous time signal equation:

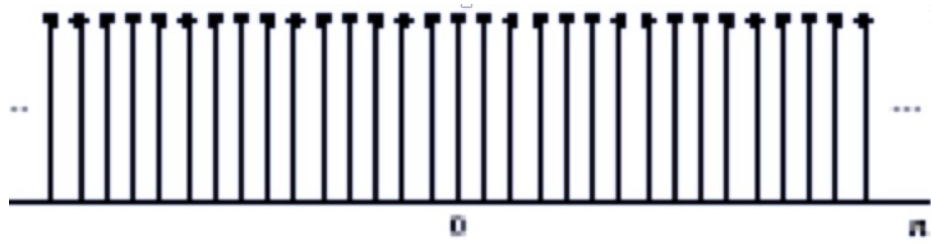
$$x(t) = A \cos(\omega_0 t + \phi) \quad \xrightarrow{\text{sampling}} \quad x[n] = A \cos(\Omega_0 n + \phi)$$

- ◆ The discrete time signal is sampled at f_s , where $T_s = 1/f_s$ is the sampling period (i.e. time step between successive samples).
- ◆ Note that Ω_0 in discrete time domain is angle increment of this sinusoidal signal between samples. Its unit is radians/sample (not rad/sec as in continuous time case).

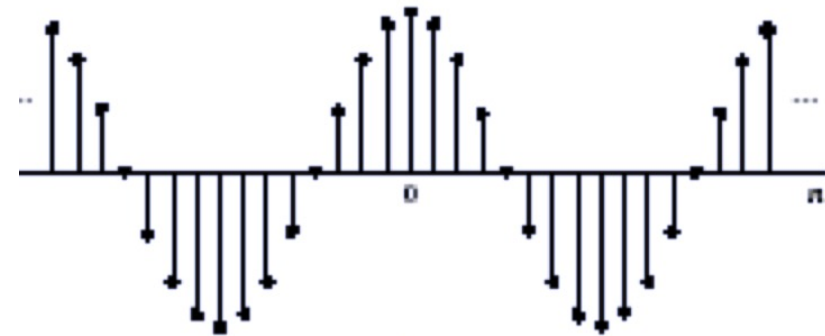
Discrete sinusoidal signal

$$x[n] = A \cos(\Omega_0 n + \phi) \quad \text{for } n = \text{integer}$$

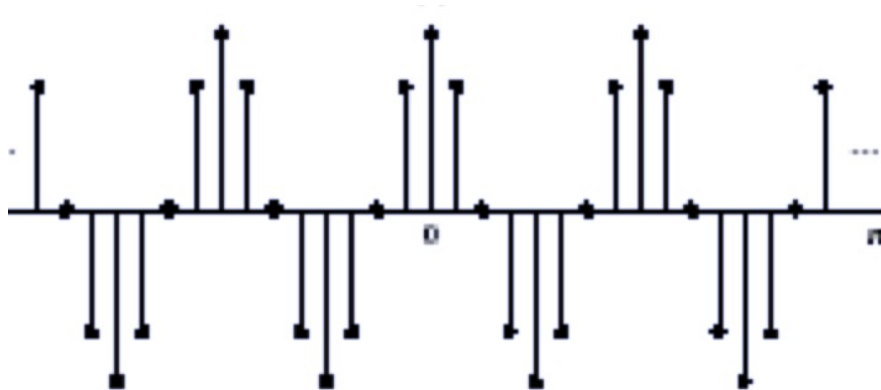
Ω_0 is in rad/sample



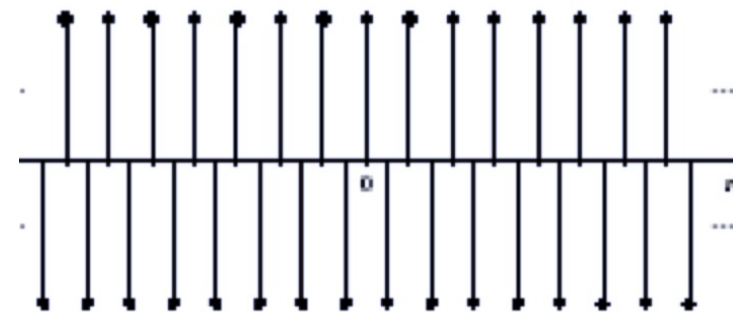
(a) When $\Omega_0 = 0$ or $x[n]$ is DC



(b) When $\Omega_0 = \frac{\pi}{8}$
i.e. 16 sample/cycle



(c) When $\Omega_0 = \frac{\pi}{4}$
i.e. 8 sample/cycle



(d) When $\Omega_0 = \pi$
i.e. 2 sample/cycle

Operations on discrete signals

- ◆ Sum of two signals:

$$s[n] = x[n] + y[n]$$

- ◆ Product of two signals:

$$p[n] = x[n] \cdot y[n]$$

- ◆ Amplification of a signal:

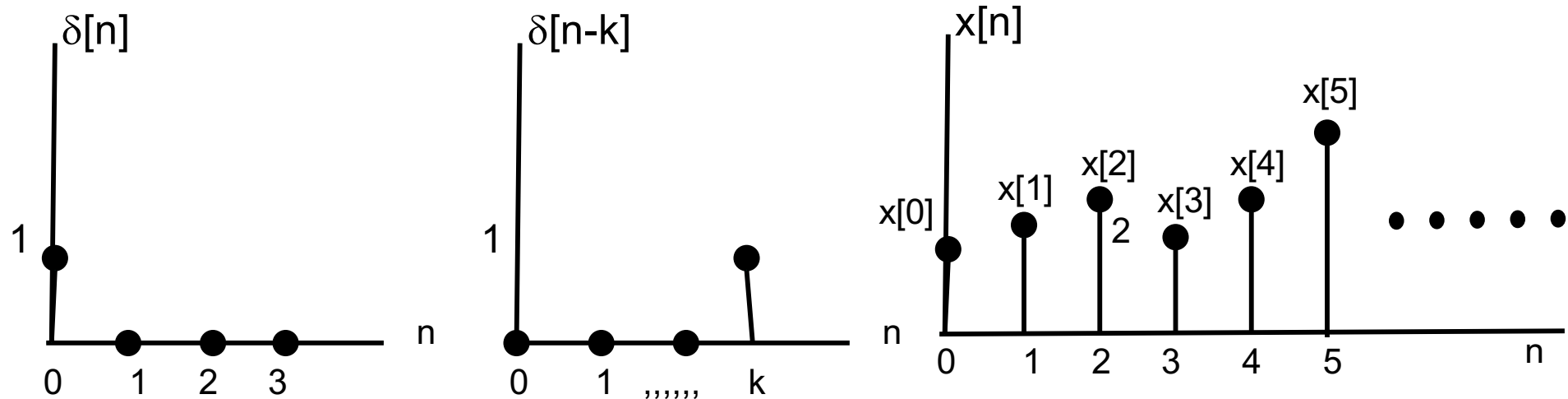
$$y[n] = \alpha \cdot x[n]$$

- ◆ Delaying a signal by k samples:

$$y[n] = x[n - k]$$

Discrete signal and impulses

- ◆ We can represent a causal discrete signal $x[n]$ in terms of sum of weighted delayed impulses:

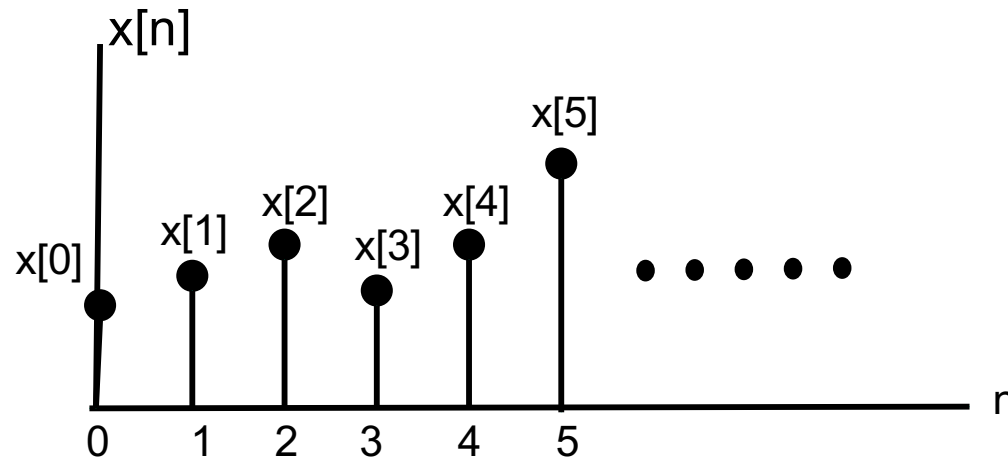


$$x[n] = x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + x[3] \delta[n - 3] + \dots$$

$$x[n] = \sum_{k=0}^{\infty} x[k] \delta[n - k]$$

Energy of a discrete signal

- ◆ The energy of a discrete signal can be computed easily – simply sum the square of each sample values:



$$E\{x[n]\} = \sum_{k=0}^{\infty} |x[k]|^2$$

- ◆ Instantaneous energy of the signal at sample i over a window of K samples is:

$$E\{x_i[n]\} = \sum_{k=0}^{K-1} |x[i+k]|^2$$

An alternative representation of discrete signals

- ◆ Instead of representing discrete signals in terms of impulse functions with various delay, we can transform the discrete signal into another domain (or mathematical representation).
- ◆ Let us assume that we use a transformation that maps an impulse function with delay k such that:

$$\delta[n - k] \xrightarrow{z} z^{-k}$$

- ◆ Then the discrete signal $x[n]$ is transformed to another function in terms of the variable z :

$$x[n] = x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + x[3] \delta[n - 3] + \dots$$

$$x[n] \xrightarrow{z} X[z] = x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \dots$$

$$X[z] = \sum_{k=0}^{\infty} x[k] z^{-k}$$

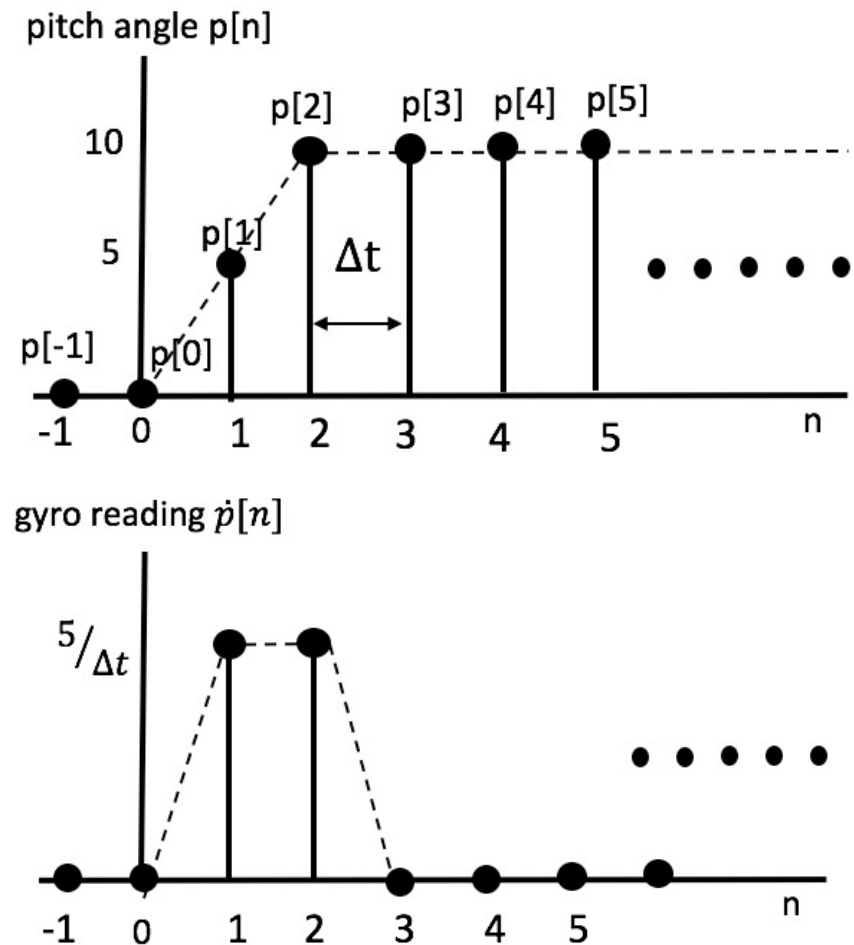
- ◆ $X[z]$ is the z -transform of the signal $x[n]$. For now, you only need to remember that z^{-k} represents k sample period delay.

Example – Gyroscope signal

- ◆ Assume pitch angle $p(t)$ changes from 0 to 10 shortly after $t = 0$
- ◆ After sampling, we get $p[n]$ as shown where it takes two sample periods to reach final value of 10
- ◆ Gyroscope measure angular velocity $\frac{dp}{dt}$
- ◆ In discrete time domain, we get

$$\frac{\Delta p}{\Delta t} = \dot{p}[n] = \frac{p[n] - p[n-1]}{\Delta t}$$

- ◆ For discrete signals, we compute differentiation using differences
- ◆ This graph shows the sample values of the gyroscope reading

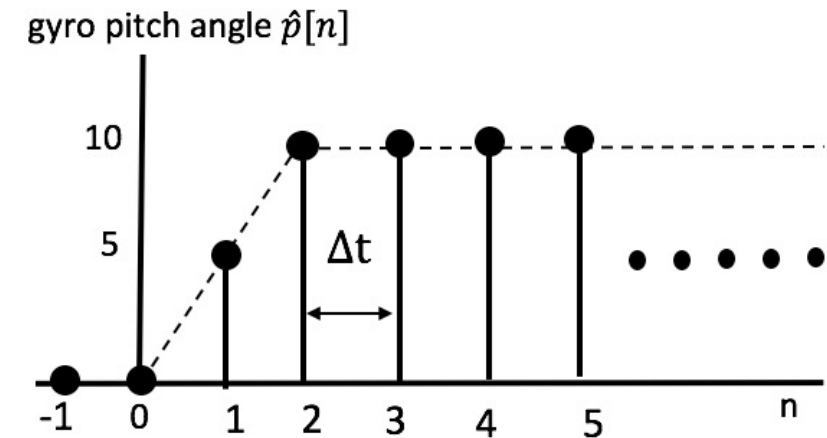
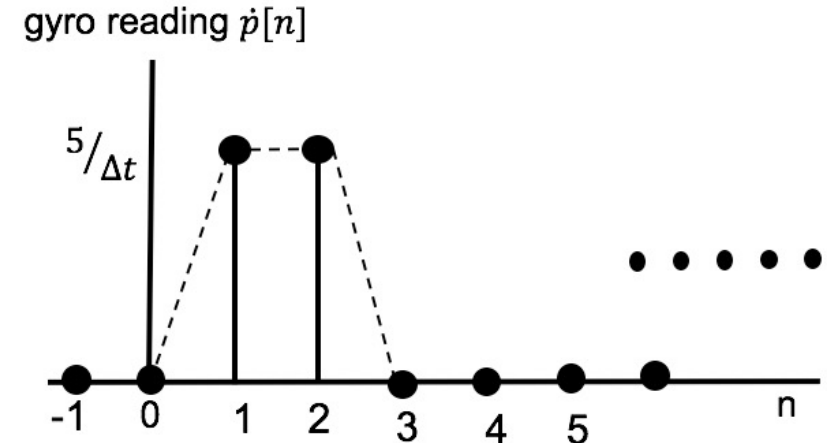


Derive pitch angle from gyro data

- ◆ How do we obtain an estimate of the pitch angle $\hat{p}[n]$ from the gyro reading $\dot{p}[n]$?
- ◆ We perform integration:

$$\hat{p}[n] = \hat{p}[n - 1] + \dot{p}[n]\Delta t$$

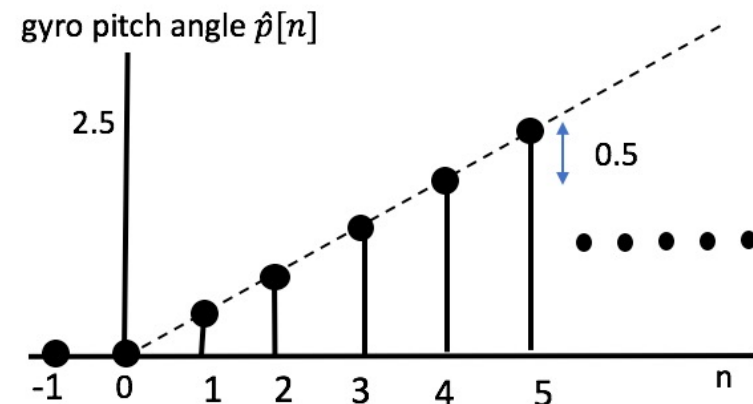
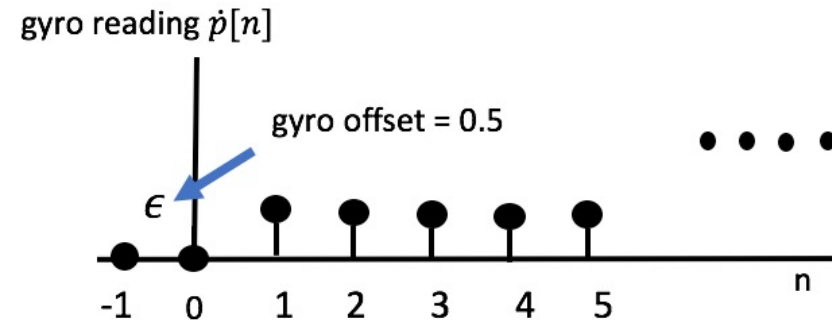
- ◆ So integration in discrete time domain is performed with summation in a **recursive equation**



Problem of Drift in Gyroscope

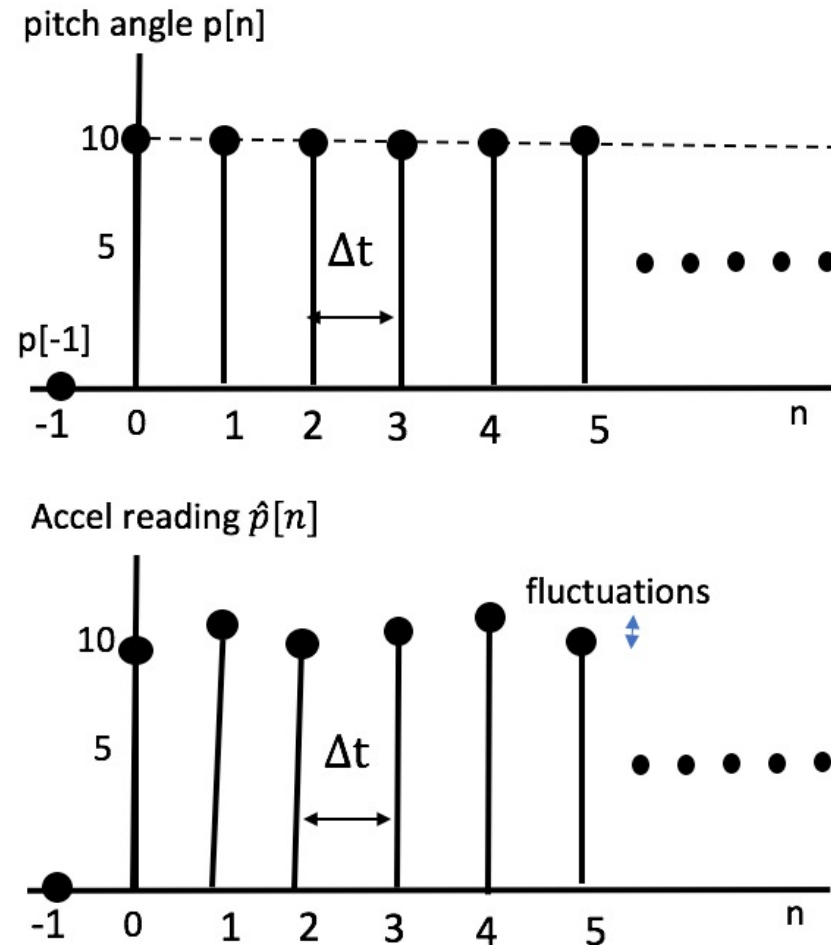
- ◆ All transducers that measure physical quantities (such as angular velocity) has errors
- ◆ In gyroscope, the problematic error is the DC offset. That means even if the gyro is NOT rotating, the IMU returns a small value ϵ
- ◆ The result of such offset after integration is to yield a pitch angle estimate that increases or decrease linearly with time as shown here.

$$\hat{p}[n] = \hat{p}[n - 1] + \epsilon \quad \text{for } n > 0$$



Problem of noise in accelerometer data

- ◆ Let us assume that the Pybench board is instantaneously rotated to 10 degrees
- ◆ The accelerometer's measurement will reflect this, but superimpose on this is a lot of noise
- ◆ Partly is due to unwanted forces acting on the mass inside the IMU



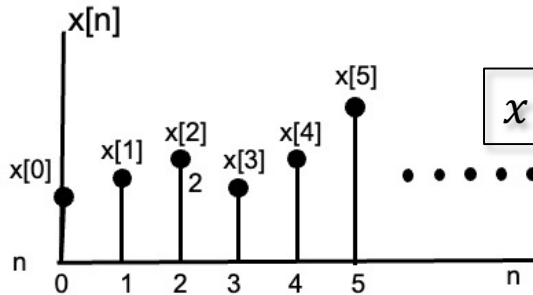
Three Big Ideas (1)

1. Discrete sinusoidal signal is of the form:

$$x(t) = A \cos(\omega_0 t + \phi) \quad \xrightarrow{\text{sampling}} \quad x[n] = A \cos(\Omega_0 n + \phi)$$

Ω_0 is angle increment between samples
In radian/sample

2. Any discrete signal can be expressed as **weighted sum** of unit impulses, delayed and scaled.



$$x[n] = x[0] \delta[n] + x[1] \delta[n - 1] + x[2] \delta[n - 2] + x[3] \delta[n - 3] + \dots$$

Three Big Ideas (2)

3. If we map the delayed impulse (delay function) as:

$$\delta[n - k] \rightarrow z^{-k}$$

We transform discrete time signals to a new domain, called z-domain.

This transform is known as z-transform, and is useful to model discrete signals.

$$x[n] \xrightarrow{\text{z-transform}} X[z] = x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \dots$$