

Lecture 12

Discrete Time Systems

Prof Peter YK Cheung

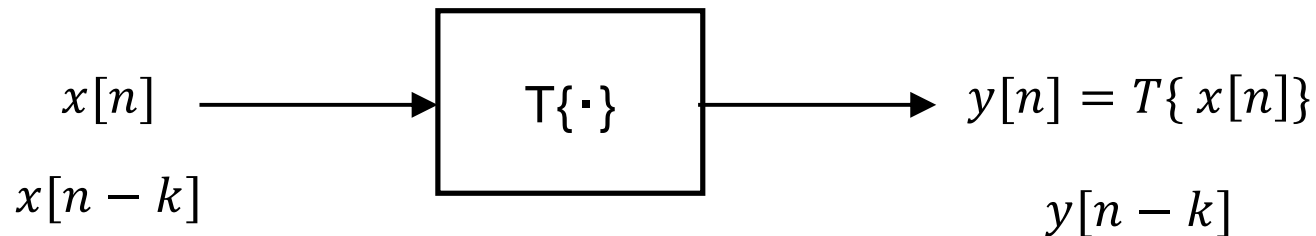
Dyson School of Design Engineering



URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

Linear Discrete time systems

- ◆ A discrete time system takes in a sequence of discrete values $x[n]$ at the input and produces an output sequence $y[n]$ through some internal operation or transformation $T\{\cdot\}$



- ◆ The system is LINEAR if it obeys the principle of superposition:

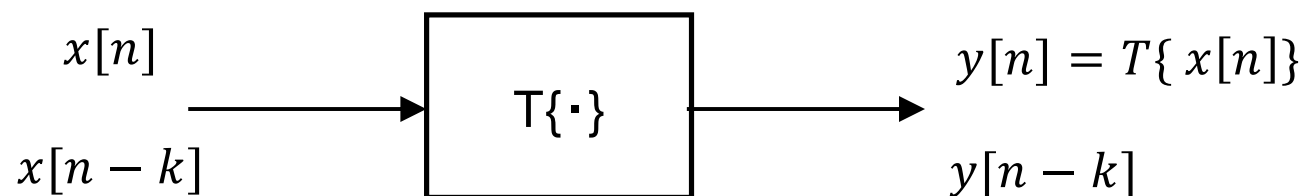
$$y[n] = T\{a_1x_1[n] + a_2x_2[n]\} = a_1T\{x_1[n]\} + a_2T\{x_2[n]\}$$

- ◆ The system is shift-invariant if:

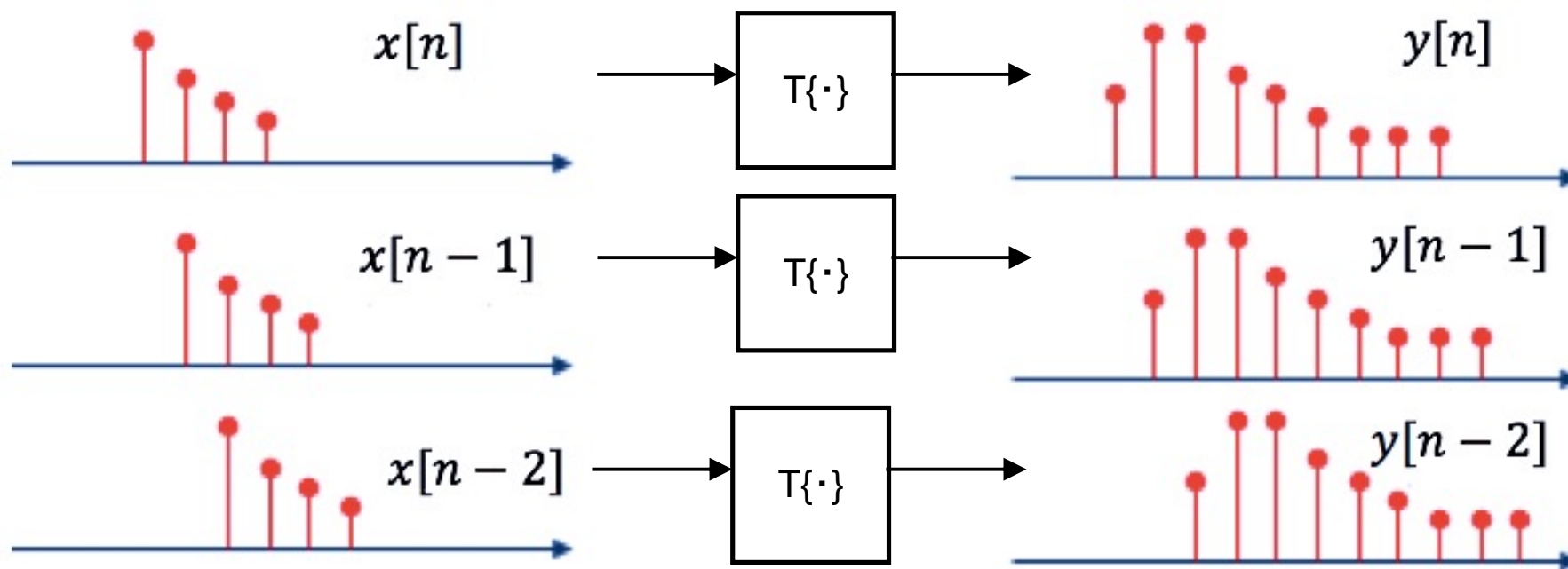
$$\begin{aligned} y[n] &= T\{a_1x_1[n-k] + a_2x_2[n-k]\} \\ &= y[n-k] \end{aligned}$$

Shift-invariant Discrete time systems

- ◆ Furthermore, a system is shift-invariant if delaying the input $x[n]$ by k samples results in the same output $y[n]$, but delayed also by k .

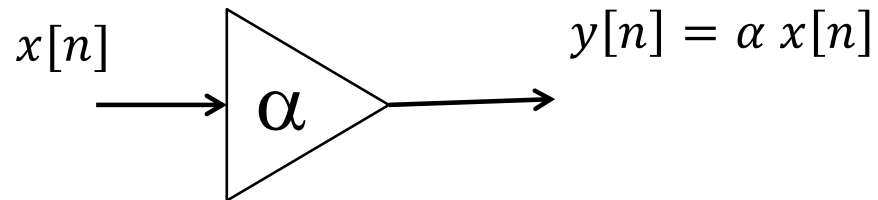


- ◆ In this course, we only consider linear shift-invariant discrete systems.

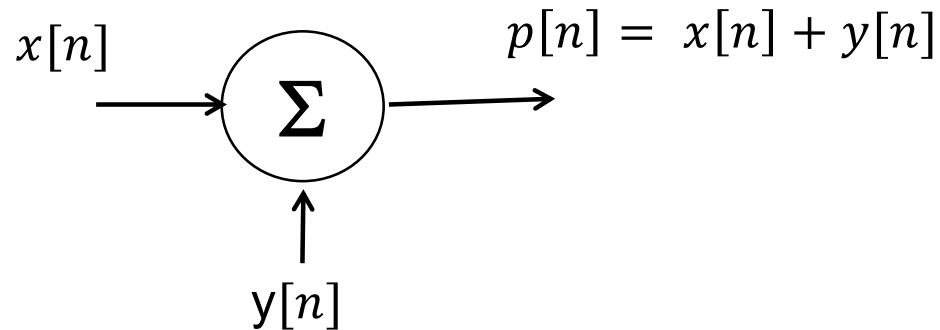


Basic building blocks in a discrete linear system

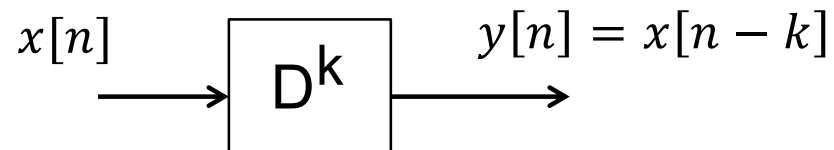
◆ Scaling



◆ Adding

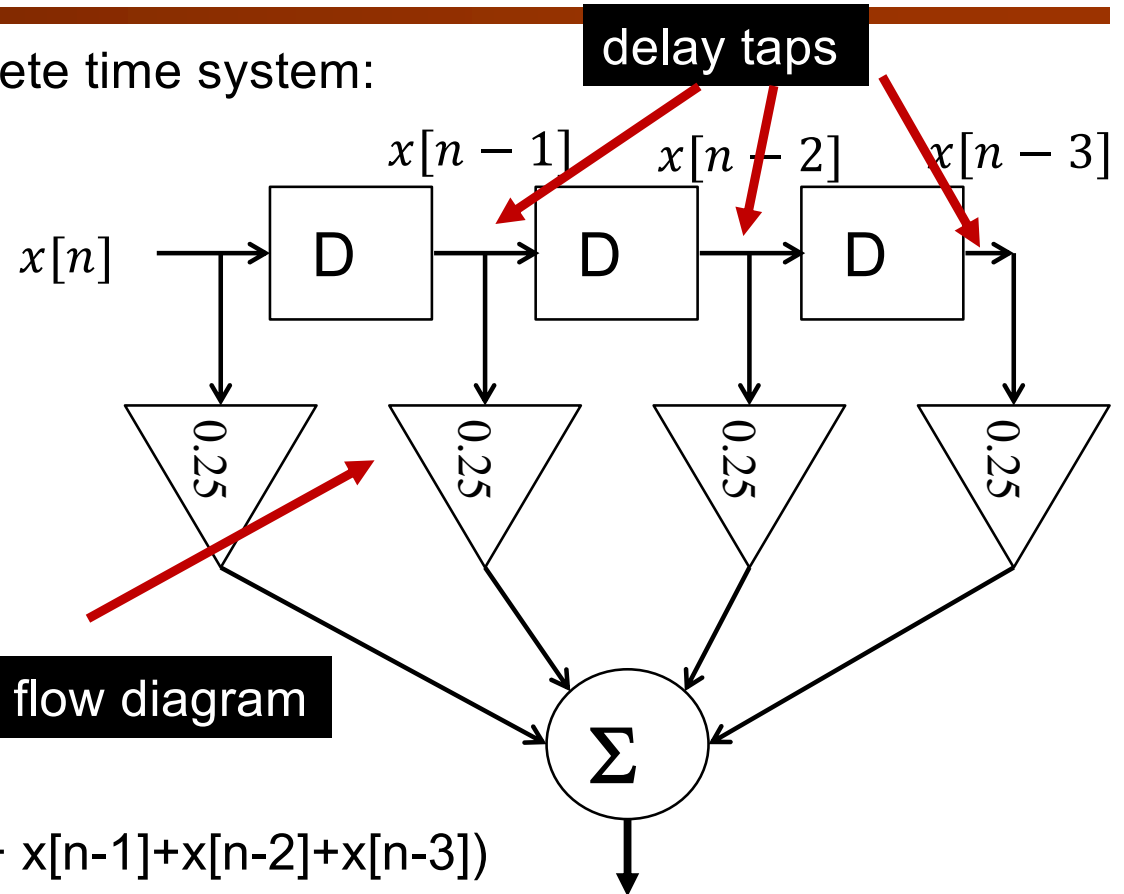
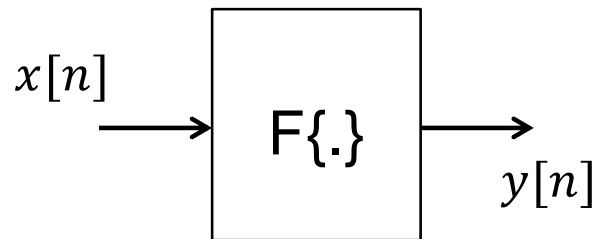


◆ Delay (i.e. D^k = time shift by k sample periods)



Moving average filter

- ◆ Consider the following discrete time system:



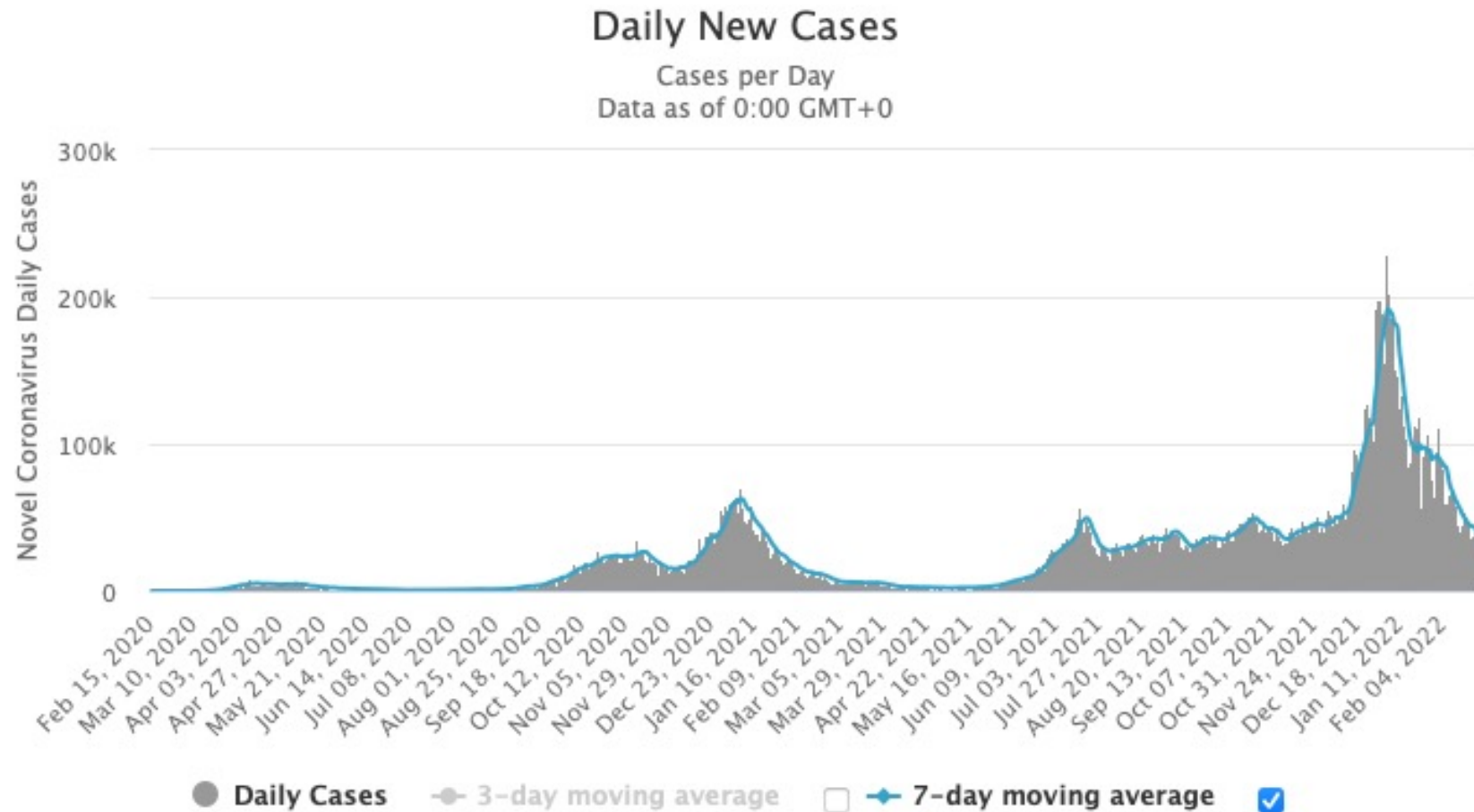
difference equation

$$y[n] = 0.25 (x[n] + x[n-1] + x[n-2] + x[n-3])$$

- ◆ This system take the current and the previous 3 input samples, and average them. This is also known as a **moving average filter**.

Example - COVID cases in UK

Daily New Cases in the United Kingdom



z-transform and difference equation

- ◆ According to Lecture 11 slide 9, if the z-transform of $x[n]$ is $X[z]$:

then,

$$x[n] \xrightarrow{Z} X[z]$$
$$x[n - k] \xrightarrow{Z} X[z] z^{-k}$$

- ◆ In other words, delaying a signal $x[n]$ by k sample period is equivalent to multiplying its z-transform $X[z]$ with z^{-k} .
- ◆ We can apply this important property of z-transform (known as the **shift property**) to the difference equation relating the input sequence to the output sequence:

Difference equation $\rightarrow y[n] = 0.25 (x[n] + x[n-1] + x[n-2] + x[n-3])$

$$Y[z] = 0.25\{X[z] + X[z]z^{-1} + X[z]z^{-2} + X[z]z^{-3}\}$$

z-transform equation $\rightarrow Y[z] = 0.25(1 + z^{-1} + z^{-2} + z^{-3})X[z]$

- ◆ This is the z-domain version of the difference equation in terms of z^{-k} , where k is delay in unit of sample.

Transfer function in the z-domain

- ◆ Take the results from the previous slide and re-arrange:

$$y[n] = 0.25 (x[n] + x[n-1] + x[n-2] + x[n-3])$$

$$Y[z] = 0.25 \{X[z] + X[z]z^{-1} + X[z]z^{-2} + X[z]z^{-3}\}$$

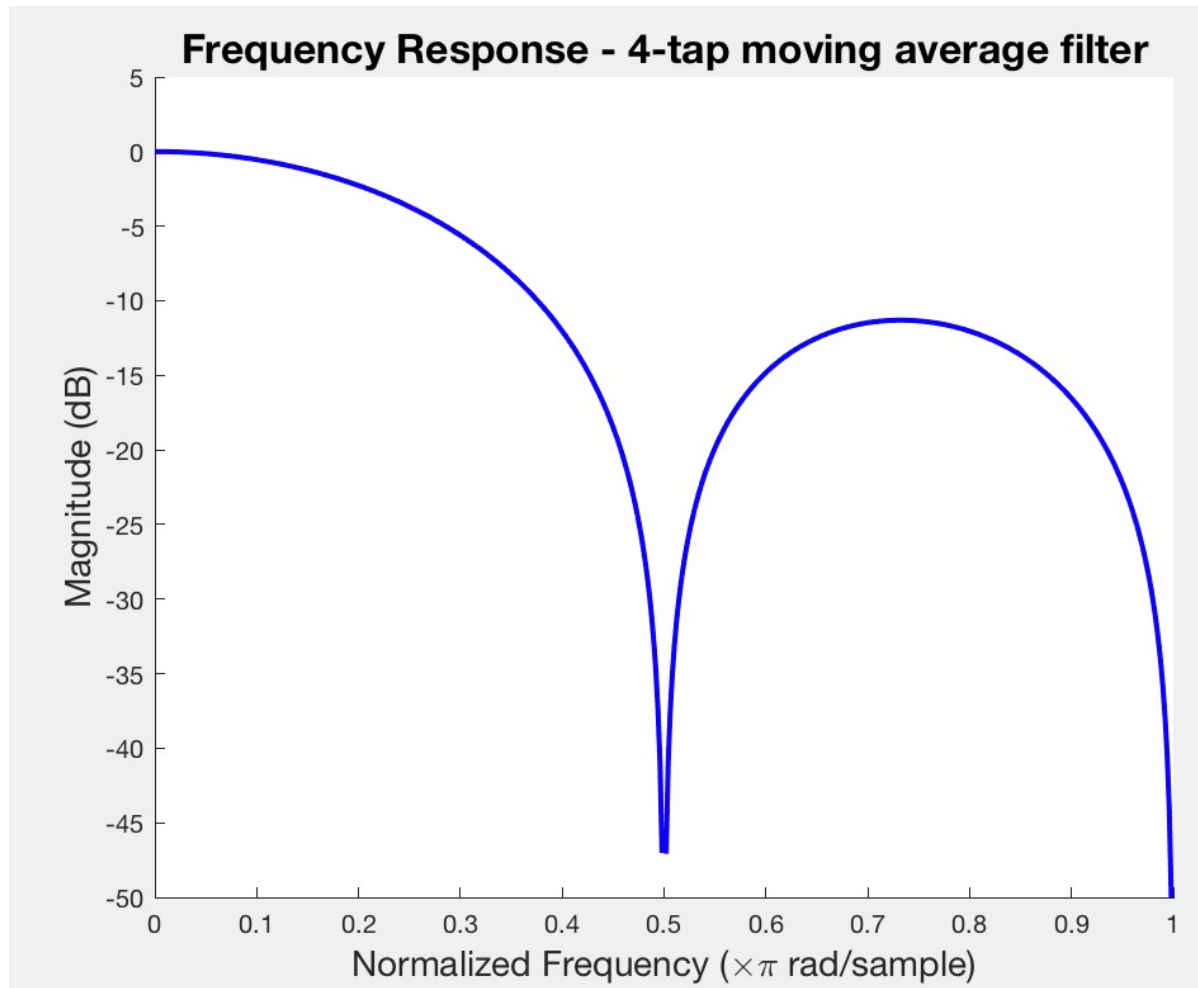
$$Y[z] = 0.25(1 + z^{-1} + z^{-2} + z^{-3})X[z]$$

$$H[z] = Y[z]/X[z] = 0.25(1 + z^{-1} + z^{-2} + z^{-3})$$

- ◆ As in the case of Laplace transform, in the z-domain,
transfer function $H[z]$ = output $Y[z]$ / input $X[z]$
- ◆ This moving average filter takes the average of the current data sample $x[i]$, and the previous three samples $x[i-1]$, $x[i-2]$ and $x[i-3]$, to produce the output $y[i]$.
- ◆ The averaging function has a **smoothing effect** – that is, it performs the function of a **lowpass filter**.

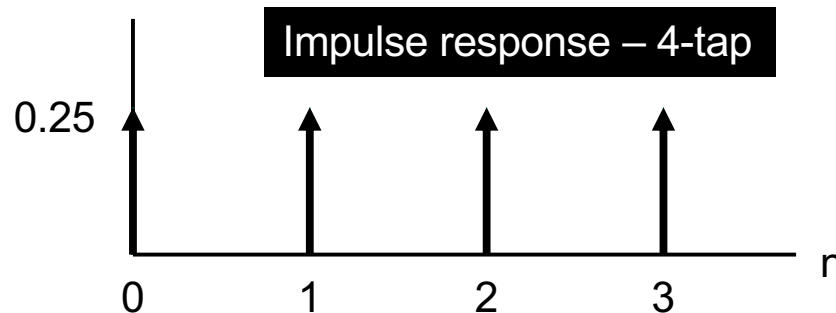
Frequency Response of this filter

- ◆ Here is the frequency response of this moving average filter:

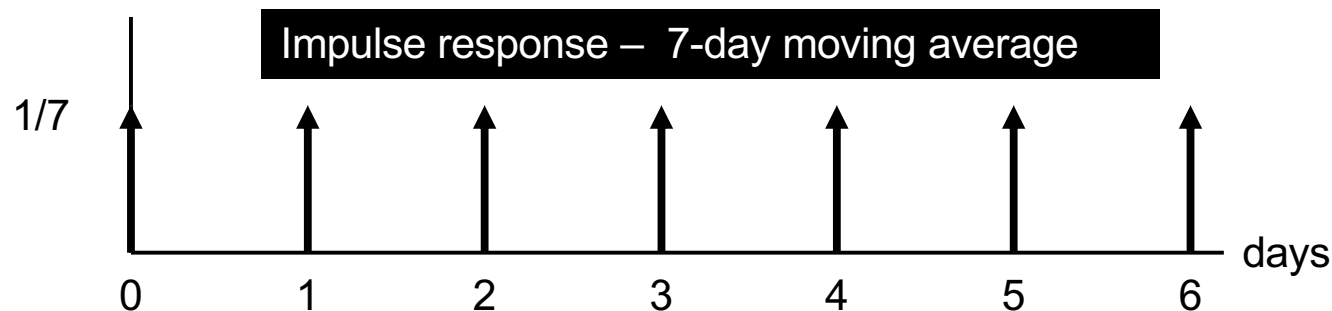


Impulse Response of the moving average filter

- ◆ The four-tap moving average filter has an impulse response of:



- ◆ The reason is that an impulse at $n = 0$ will have an impact on the output of $\frac{1}{4}$ on the current sample time, and the three subsequent sample times.
- ◆ For the 7-days moving average filter, the impulse response is:



General FIR filters

- ◆ Instead of using the same coefficient values in the moving average filter, one could use different coefficients at different delay taps.
- ◆ The number of delay taps can be increased to N.
- ◆ This will implement a filter function of the form as difference equation:

$$y[n] = b_0x[n] + b_1x[n - 1] + b_2x[n - 2] + \dots + b_{N-1}x[n - (N - 1)]$$

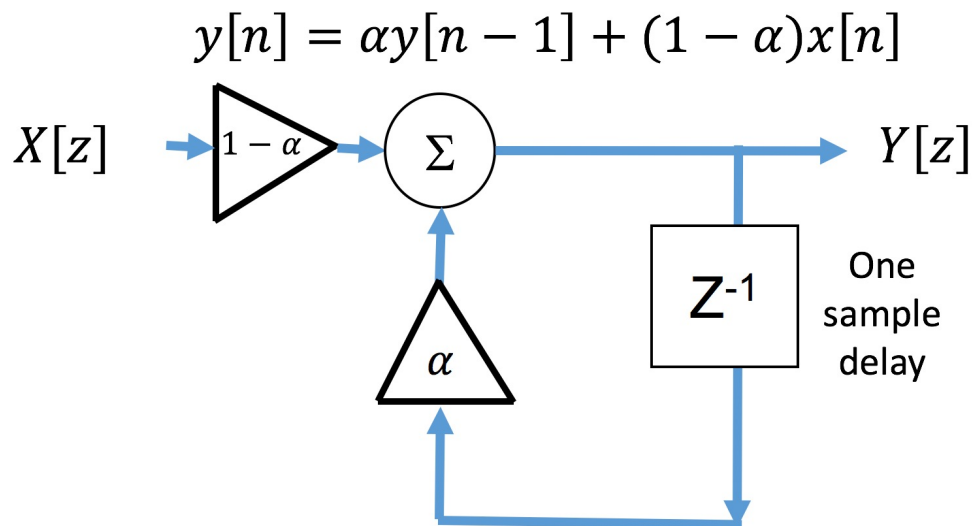
- ◆ In z-domain form:

$$Y[z] = (b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \dots + b_{N-1}z^{-(N-1)})X[z] = X[z] \sum_{k=0}^{N-1} b_k z^{-k}$$
$$H[z] = \frac{Y[z]}{X[z]} = b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} + \dots + b_{N-1}z^{-(N-1)} = \sum_{k=0}^{N-1} b_k z^{-k}$$

- ◆ By choosing different coefficients $b_0, b_1, b_2, \dots, b_{N-1}$, one can implement different types of filters: **lowpass**, **bandpass**, **highpass** etc.
- ◆ Such a filter will have N terms in the impulse response, where N is the number of signal taps $x[n], \dots, x[n-(N-1)]$. Therefore it is also known as a **finite impulse response** filter (FIR) of order **N**.

Recursive Filter

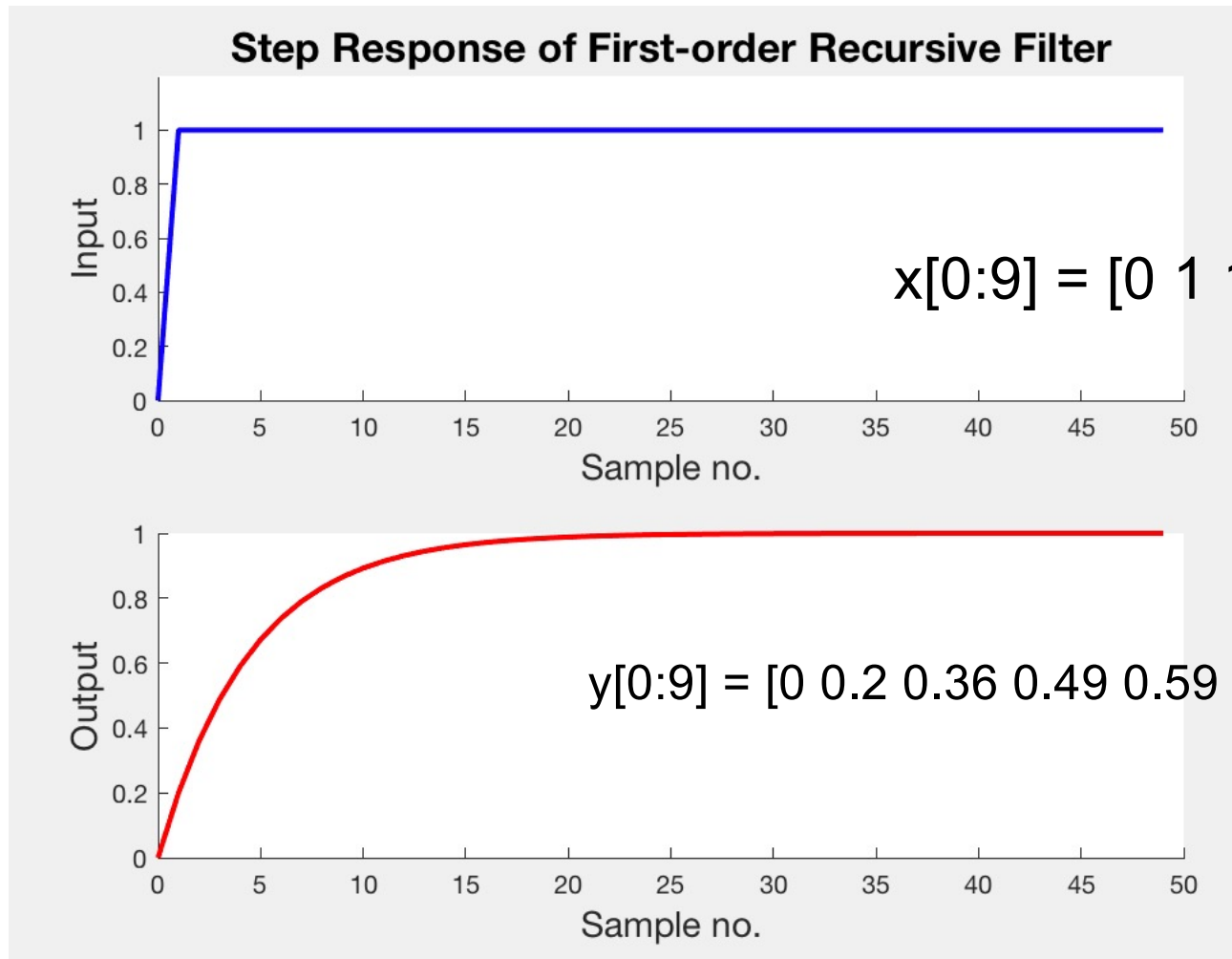
- ◆ **FIR filters** derives the current output from **current** and **previous** inputs
- ◆ Such a filter does not make use of previous outputs – that is, it does not rely on past information
- ◆ **Recursive filter** or **IIR filter** is different – it derives the current output from **both input and previous output** samples.
- ◆ Here is one of the simplest recursive filter:



$$\begin{aligned} Y[z] &= \alpha Y[z] z^{-1} + (1 - \alpha)X[z] \\ \Rightarrow Y[z] - \alpha Y[z] z^{-1} &= (1 - \alpha)X[z] \\ \Rightarrow (1 - \alpha z^{-1})Y[z] &= (1 - \alpha)X[z] \\ \Rightarrow H[z] &= \frac{Y[z]}{X[z]} = \frac{(1 - \alpha)}{(1 - \alpha z^{-1})} \end{aligned}$$

Step response of Recursive Filter

- ◆ Let us consider the response of this filter to an impulse at the input



$$x[0:9] = [0 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]$$

$$y[0:9] = [0 \ 0.2 \ 0.36 \ 0.49 \ 0.59 \ 0.67 \ 0.74 \ 0.79 \ 0.83 \ 0.87]$$

Three Big Ideas

1. A discrete-time system can be represented in three ways:

- Block diagram

- Difference equation, e.g. $y[n] = 0.25 (x[n] + x[n-1] + x[n-2] + x[n-3])$

- Impulse response in the z-domain, e.g. $H[z] = \frac{1}{4} (1 + z^{-1} + z^{-2} + z^{-3})$

2. A Finite Impulse Response filter (FIR), as the name implies, is one where the effect of input signal decay to zero after finite time samples, while Infinite Impulse Response (IIR) or recursive filters are one where the effect of input signal lasts forever (at least in theory).

3. IIR filter is much more efficient to implement than FIR filter. For the same filtering effect, IIR filter has much lower number of multiplier and add operations than FIR filter.