Lecture 13
Impulse Response & Filters

Peter Cheung
Dyson School of Design Engineering

URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk
Impulse Response of a Discrete System

◆ The response of a discrete time system to a discrete impulse at the input is known as the system’s **impulse response**

\[ x[n] = \delta[n] \quad h[n] = \text{discrete impulse response} \]

\[ x(t) = \delta(t) \quad h(t) = \text{continuous impulse response} \]

◆ The impulse response of a **linear** system completely defines and characterises the system – both its transient behaviour and its frequency response.

◆ This applies to both continuous time and discrete time linear systems.

◆ An impulse signal contains ALL frequency components (L3, S7). Therefore, applying an impulse to input stimulates the systems at ALL frequencies.

◆ Since integrating a unit impulse = a unit step function, we can obtain the step response of the system by integrating the impulse response.
Remember from L10, S7, we can represent a causal discrete signal $x[n]$ in terms of sum of weighted delayed impulses:

\[
x[n] = \sum_{k=0}^{\infty} x[k] \delta[n - k]
\]

**Discrete signal as sum of weighted impulses**
Impulse Response and Convolution

◆ We can therefore derive the output of a discrete linear system, by adding together the system’s response to EACH input sample separately.

◆ This operation is known as \textit{convolution}:

\[
y[n] = x[n] * h[n] = \sum_{m=0}^{\infty} x[m]h[n - m]
\]
Convolution example - COVID vaccination

- Covid vaccine require two doses, 3 weeks apart. Impulse response $h[n]$ for this vaccination system is:

  ![Impulse response diagram]

- UK started vaccination of its elderly population with a plan of vaccinating, say, 1000 people per day after the first day (day 0). The input $x[n]$ is:

  ![Input signal diagram]

- How many doses would NHS need to provide from day 0? (i.e. $y[n]$)
Graphical representation of Convolution

Convolution in 4 steps:
1. Reflect impulse response at origin
2. Multiply with input sequence
3. Sum to produce current output
4. Shift 1 sample period and repeat
Since a unit impulse contains all frequency, and its Fourier transform is a constant at 1 (see L3, S7), the Fourier transform of the impulse response $h[n]$ or $h(t)$ give us the systems’ frequency response:

$H[.]$ 

This applies to both continuous time and discrete time linear systems.

An impulse signal contains ALL frequency components (L3, S7). Therefore, applying an impulse to input stimulates the systems at ALL frequencies.
Moving Average Filter = FIR lowpass filter

- N-tap (or N point) moving after filter – high N, lower the cut off frequency
- For a N-tap moving average filter, it impulse response has N impulses.
- If input x[n] has M non-zero samples (i.e. finite length), output y[n] is also finite in length, and has M+N non-zero samples. Hence the name Finite Impulse Response (FIR) filter.
Frequency Response of N-tap moving average filter

- The impulse response of a moving average filter is a rectangular pulse.
- The Fourier transform of a rectangular pulse is of the form \((\sin x)/x\) or \(\text{sinc}(x)\) function (see Lecture 3 slide 6) in the case of continuous time.
- For discrete time case, the frequency response of a moving average filter with N taps (or points) is:
  \[
  H[f] = \frac{\sin(fN)}{N \sin f}
  \]
- Here \(f\) is normalised to 0 -> 0.5 x sampling frequency \(fs\).
Recursive or Infinite Impulse Response Filter

FIR

\[ x[n] = \{1.0, 1.0, 1.0, 1.0, 1.1, 0.8, 1.2, 0.9, 1.0, 1.2, 0.9, \ldots\} \]

\[ y[n] = (0.25, 0.5, 0.75, 1.0, 1.025, 0.975, 1.025, 1.0, 1.0, 1.025, 1.0, \ldots) \]

IIR

\[ x[n] \]

\[ y[n] = (0.4, 0.64, 0.784, 0.87, 0.962, 0.897, 1.018, 0.971, 0.983, 1.07, 1.002, \ldots) \]
Complementary Filter used with IMU

\[ \theta_{new} = \alpha \times \left( \theta_{old} + \dot{\theta} \, dt \right) + (1 - \alpha) \times \rho \]

where

- \( \alpha \) = scaling factor chosen by users and is typically between 0.7 and 0.98
- \( \rho \) = accelerometer angle
- \( \theta_{new} \) = new output angle
- \( \theta_{old} \) = previous output angle
- \( \dot{\theta} \) = gyroscope reading of the rate of change in angle
- \( dt \) = time interval between gyro readings
Signal flow diagram model

\[ p[n] = \alpha(p[n-1] + g[n] \times dt) + (1 - \alpha)a[n] \]

def read_imu(dt):
    global g_pitch
    alpha = 0.7  # larger = longer time constant
    pitch = int(imu.pitch())
    roll = int(imu.roll())
    g_pitch = alpha*(g_pitch + imu.get_gy()*dt*0.001) + (1-alpha)*pitch
Now assume that gyroscope reading is zero (i.e. steady state tilt), $g[n] = 0$.

Now the system is exactly the same as that in Lecture 12, slide 12.

Therefore the accelerometer data $a[n]$ is lowpass filtered!xx

\[
p[n] = \alpha p[n - 1] + (1 - \alpha) a[n]
\]

\[
H[z] = \frac{P[z]}{A[z]} = \frac{1 - \alpha}{1 - \alpha z^{-1}}
\]

Time Constant $\tau \approx \frac{\alpha}{1-\alpha} \, dt$
and $dt$ is the sampling period
Assume the gyro is not moving, but has a constant offset $\dot{\theta}_e$.

- $dt$ is also constant.

- If $\alpha = 1$, $p[n]$ is a ramp with a gradient of $\dot{\theta}_e$.

- If $\alpha < 1$, then the effect of the error overtime diminishes to $\alpha^n \to 0$.

Integrating the gyroscope reading:

$$p[n] = \alpha(p[n - 1] + \theta[n]dt)$$

$$p[n] = \alpha^n p[0] + \alpha^n k$$
Three Big Ideas (1)

1. A discrete time system can be characterized by its impulse response:
   \[ h[n] = b_0 \delta[n] + b_1 \delta[n-1] + b_2 \delta[n-2] \ldots + b_k \delta[n-k] \]

   ![Impulse response of a 4-tap moving average filter](image)

2. Once we know the impulse response \( h[n] \), and the input sequence \( x[n] \), we can find the output \( y[n] \) by convolution:
   \[ y[n] = x[n] \ast h[n] = \sum_{m=0}^{\infty} x[m]h[n-m] \]
Three Big Ideas (2)

3. Convolution operation can be performed in four steps:

1) **Reflect** impulse response at original to get h[n-m]
2) **Multiply** input sequence x[m] with h[n-m]
3) **Sum** the product of the two sequences to get one output y[n]

\[ y[n] = \sum_{m=0}^{\infty} x[m]h[n - m] \]

4) **Advance** the reflected impulse response by one sample period and repeat to get the next y[n]