Imperial College London

Lecture 6

Windowing Effects & Discrete Fourier Transform

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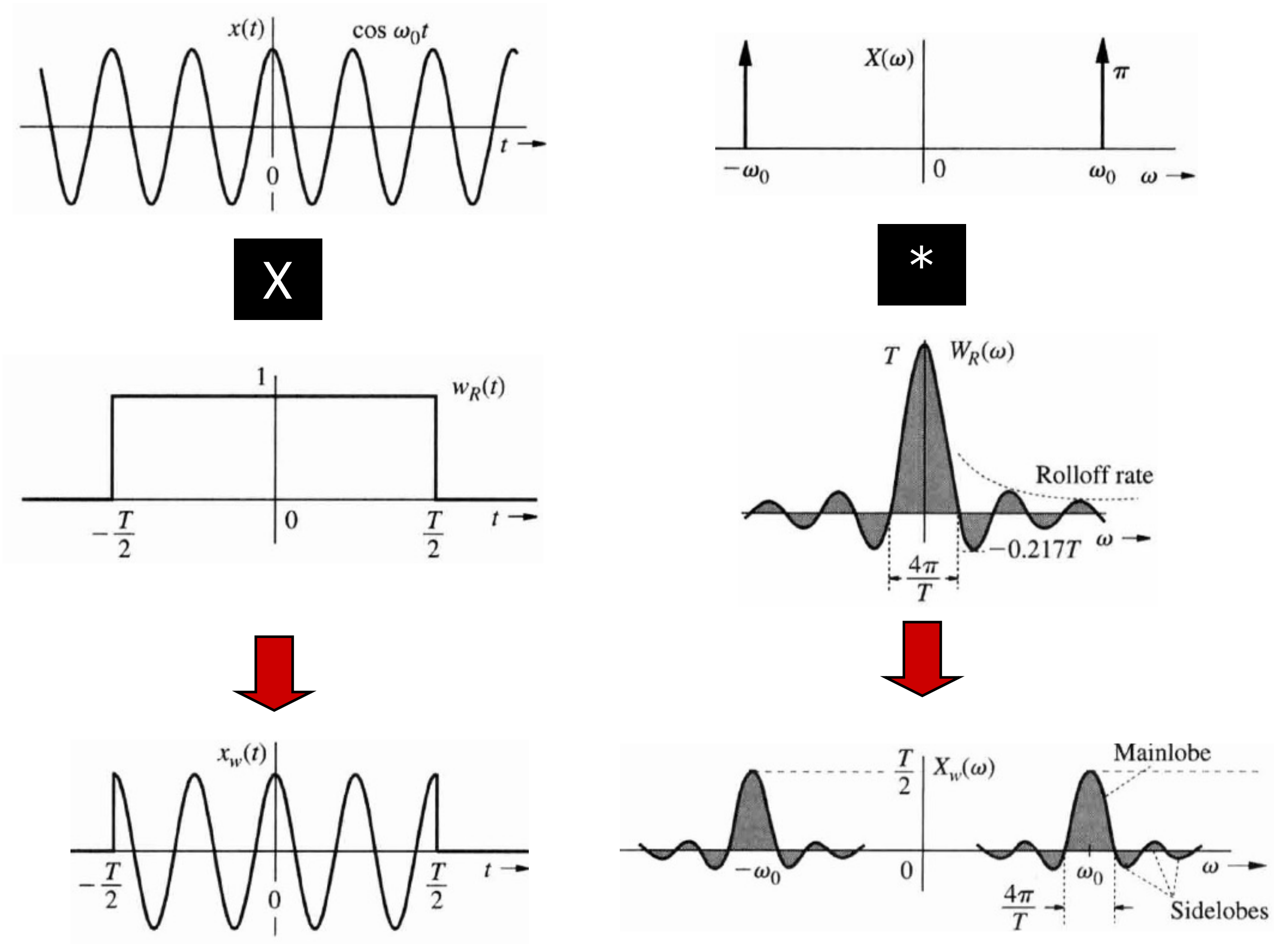
Windowing and its effect

- Extracting a segment of a signal in time is the same as multiplying the signal with a rectangular window:



Spectral spreading

Energy spread out from $\omega 0$ to width of $2\pi/T$ – reduced spectral resolution.

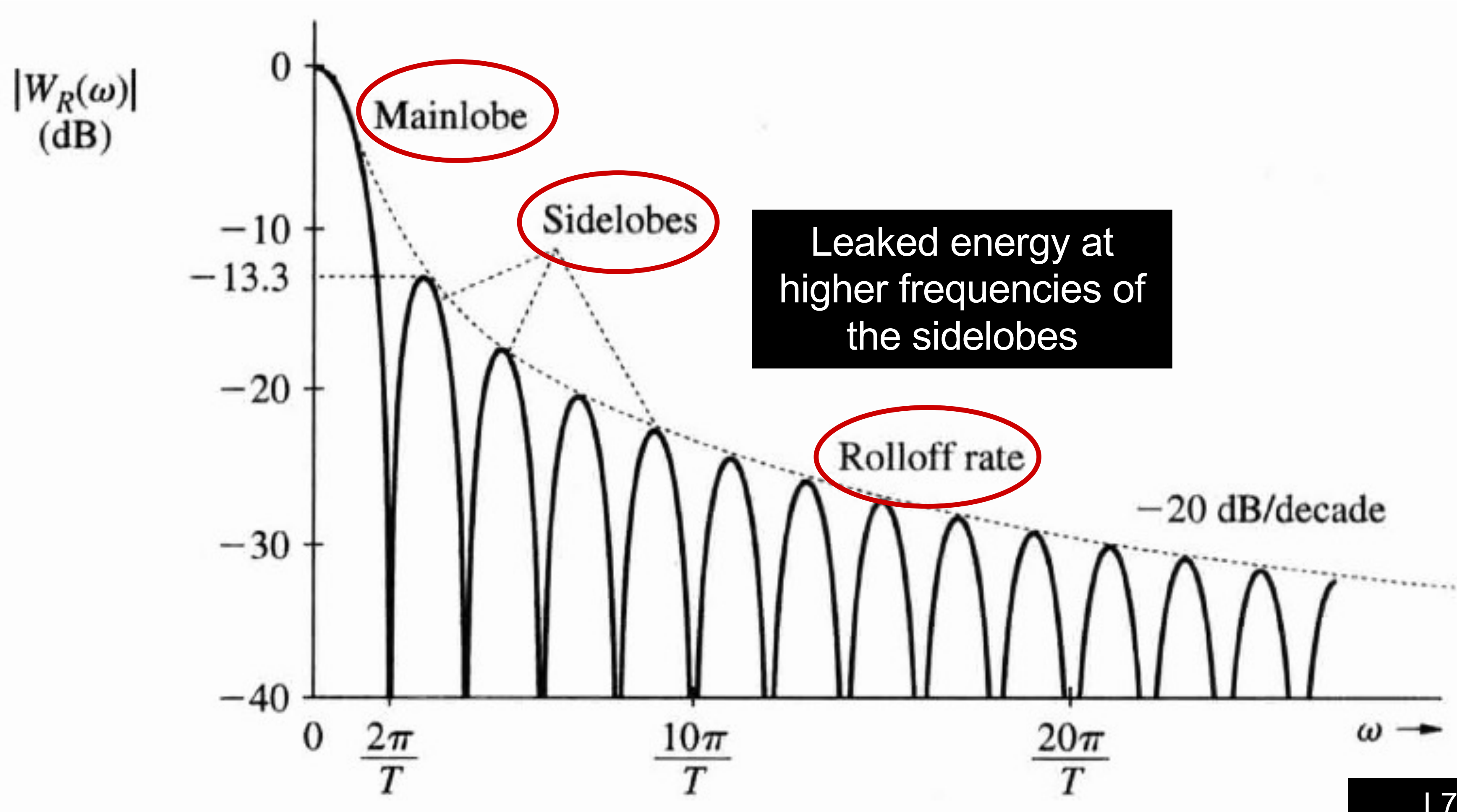
Leakage

Energy leaks out from the mainlobe to the sidelobes.

L7.8

Mainlobe & Sidelobes in dB

- Detail effects of windowing (rectangular window):



Leaked energy at higher frequencies of the sidelobes

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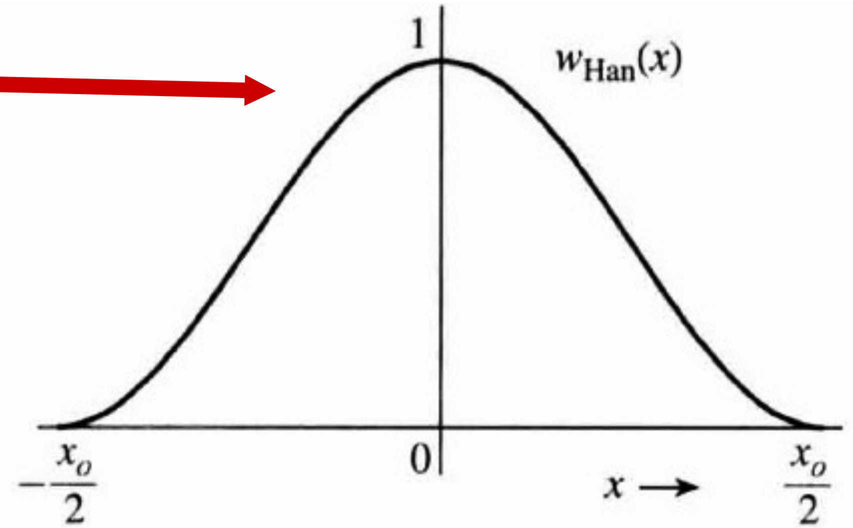
Remedies for side effects of truncation

1. Make mainlobe width as narrow as possible ➔ implies as wide a window as possible.
2. Avoid big discontinuity in the windowing function to reduce leakage (i.e. high frequency sidelobes).
3. 1) and 2) above are incompatible – therefore needs a compromise.

- Commonly replace rectangular window with one of these:
  - Hamming window
  - Hanning window
  - Barlett window
  - Blackman window
  - Kaiser window



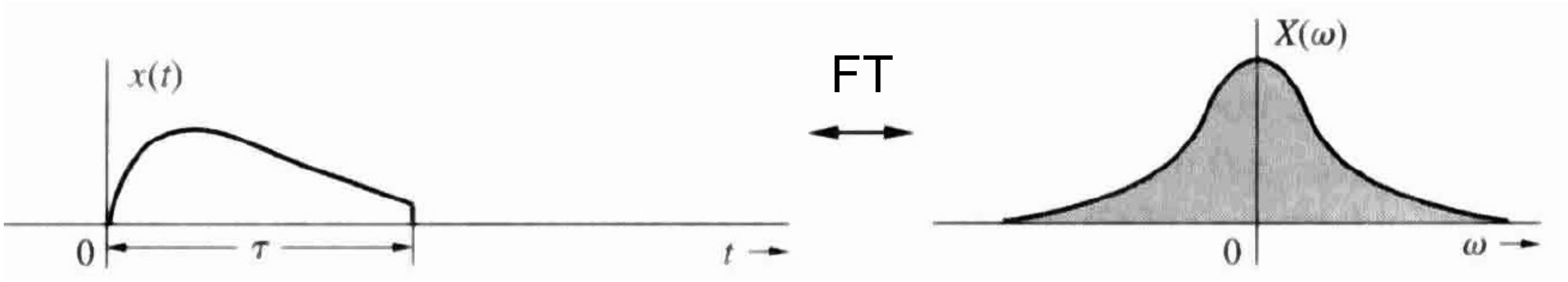
L7.8

Comparison of different windowing functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	-6	$-59.9\ (\alpha = 8.168)$

Spectral Sampling (1)

- As expected, time domain sampling has a dual: spectral sampling.
- Consider a time limited signal x(t) with a spectrum X(ω).

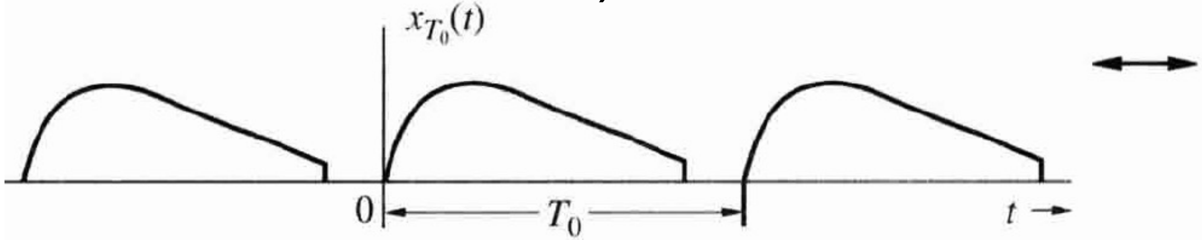


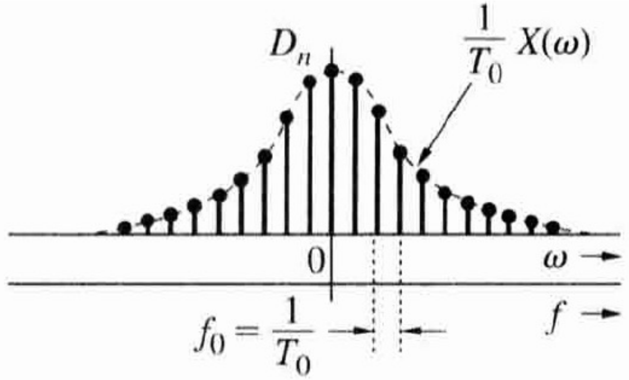
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\tau} x(t)e^{-j\omega t}dt$$

L8.4

Spectral Sampling (2)

- If we now CONSTRUCT a periodic signal $x_{T_0}(t)$, we will expect the spectrum of this signal to be discrete (expressed as Fourier series).





$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

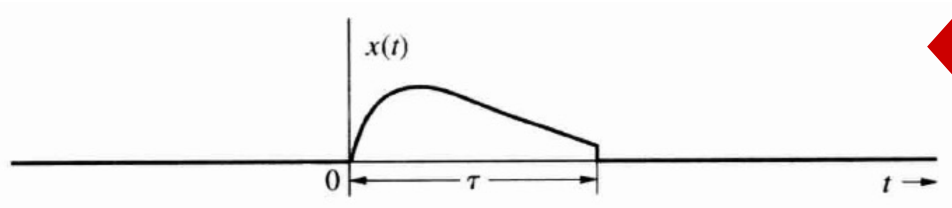
where $$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_0^{\tau} x(t) e^{-jn\omega_0 t} dt$$

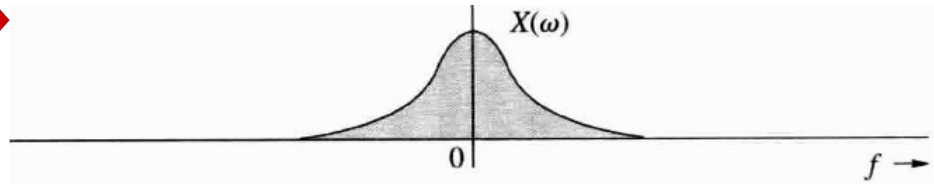
therefore $$D_n = \frac{1}{T_0} X(n\omega_0)$$

L8.4

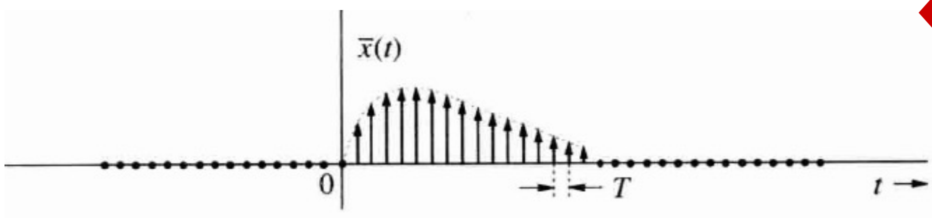
The Discrete Fourier Transform (DFT) (1)

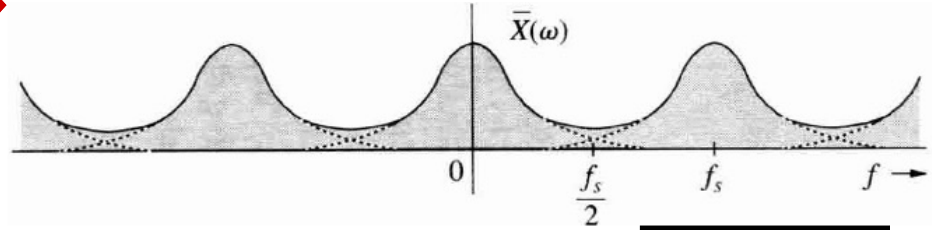
- Fourier transform is computed (on computers) using discrete techniques.
- Such numerical computation of the Fourier transform is known as Discrete Fourier Transform (DFT).
- Begin with time-limited signal x(t), we want to compute its Fourier Transform X(ω).





- We know the effect of sampling in time domain:





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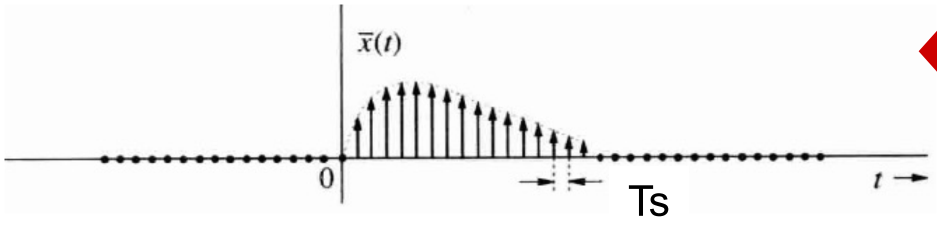
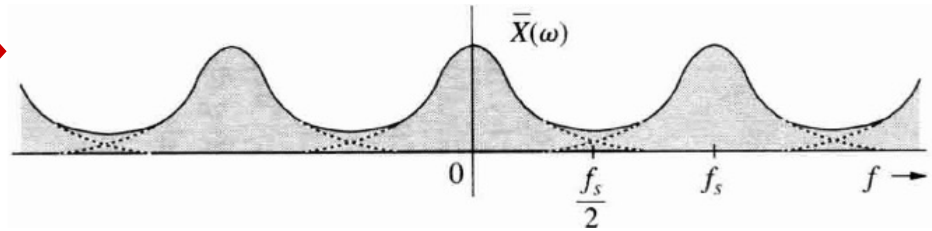
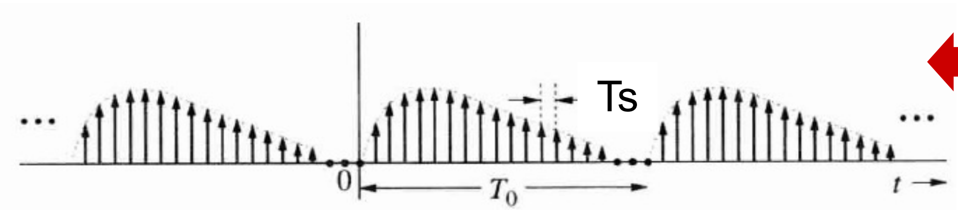
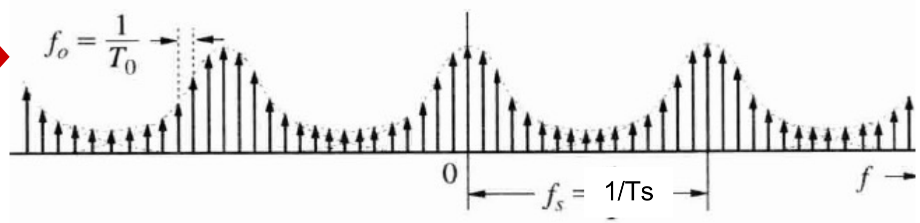
The Discrete Fourier Transform (DFT) (2)





- Now construct the sampled version of x(t) as repeated copies. The effect (from slides 6-8) is sampling the spectrum.





Number of time samples in T_0

$$N_0 = \frac{T_0}{T}$$

Number of frequency samples in f_s

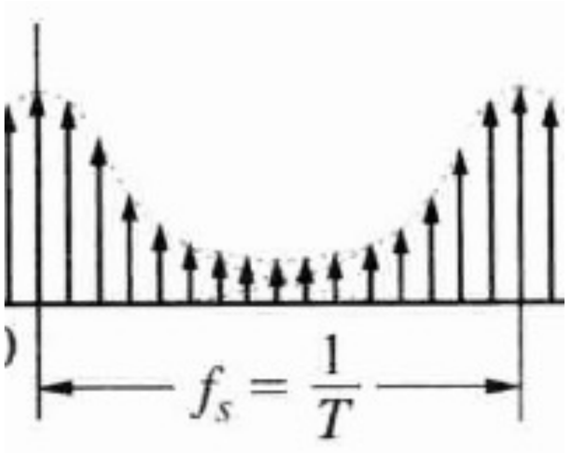
$$N_0' = \frac{f_s}{f_0}$$

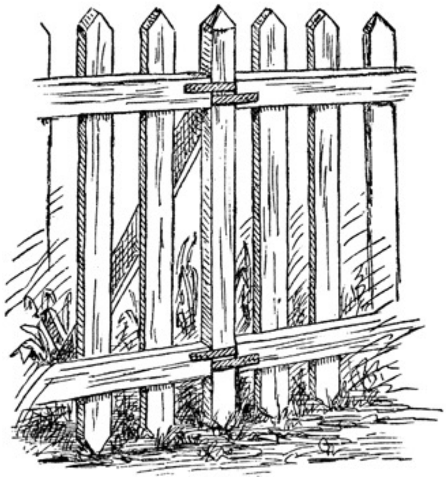
$$N_0 = \frac{T_0}{T} = \frac{1/f_0}{1/f_s} = \frac{f_s}{f_0} = N_0'$$

L8.5

Picket Fence Effect

- Numerical computation method yields uniform sampling values of $X(\omega)$.
- Information between samples in spectrum is missing – picket fence effect:





- Can improve spectral resolution by increasing number of samples used in the window N_0, i.e. the period of signal being transformed T_0.

L8.5

Formal definition of DFT

- If x[nT] and X[rω_0] are the nth and rth samples of x(t) and X(ω) respectively, then we define:

$$x_n = T \times x[nT] = \frac{T_0}{N_0} x[nT]$$ and $$X_r = X(r\omega_0)$$

where $$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

- Then

Forward DFT
$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}$$

$$\Omega_0 = \omega_0 T = \frac{2\pi}{N_0}$$

Inverse DFT
$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

L8.5

Parseval's Theorem

- The energy of a signal x(t) can be derived in time or frequency domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

- Proof:

$$x^*(t) \Longleftrightarrow X^*(-\omega)$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} x(t) x^*(t) \, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} \, d\omega \right] dt$$

Change order of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \right] d\omega$$

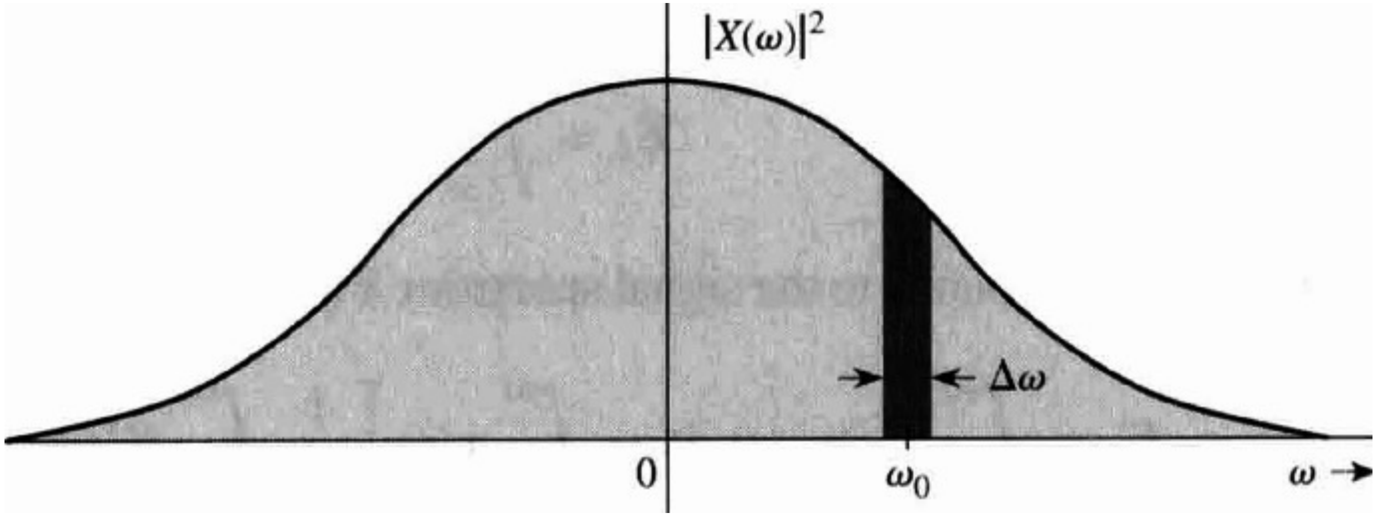
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) X^*(\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

L7.6

Energy Spectral Density of a signal

- Total energy is area under the curve of $|X(\omega)|^2$ vs ω (divided by 2π).



- The energy over a small frequency band $\Delta\omega$ ($\Delta\omega \rightarrow 0$) is:

$$\Delta E_x = \frac{1}{2\pi}|X(\omega)|^2 \Delta\omega = |X(\omega)|^2 \Delta f \qquad \frac{\Delta\omega}{2\pi} = \Delta f \text{ Hz}$$

Energy spectral density (per unit bandwidth in Hz)

L7.6

Energy Spectral Density of a REAL signal

- If x(t) is a real signal, then X(ω) and X(-ω) are conjugate:

$$|X(\omega)|^2 = X(\omega)X^*(\omega) = X(\omega)X(-\omega)$$

- This implies that X(ω) is an even function. Therefore

$$E_x = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 \, d\omega$$

- Consequently, the energy contributed by a real signal by spectral components between ω_1 and ω_2 is:

$$\Delta E_x = \frac{1}{\pi}\int_{\omega_1}^{\omega_2} |X(\omega)|^2 \, d\omega$$

L7.6

Example

- Find the energy E of signal $x(t) = e^{-at} u(t)$. Determine the frequency W (rad/s) so that the energy contributed by the spectral component from 0 to W is 95% of the total signal energy E.
- Take FT of x(t):

$$X(\omega) = \frac{1}{j\omega + a}$$

- By Parseval's theorem:

$$E_x = \frac{1}{\pi}\int_0^{\infty} |X(\omega)|^2 \, d\omega = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega^2 + a^2}\, d\omega = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_0^{\infty} = \frac{1}{2a}$$

- Energy in band 0 to W is 95% of this, therefore:

$$\frac{0.95}{2a} = \frac{1}{\pi}\int_0^{W} \frac{d\omega}{\omega^2 + a^2} = \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_0^{W} = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$$

$$\frac{0.95\pi}{2} = \tan^{-1}\frac{W}{a} \implies W = 12.706a \text{ rad/s}$$

- Note: For this signal, 95% of energy is in the lower frequency band from 0 to 12.7a rad/s or 2.02a Hz!!!

L7.6

Three Big Ideas

1. Extracting a portion of a signal can be modelled by multiplying the signal with a rectangular window. However, the sudden changes at the window boundaries modify the signal spectrum.

2. This causes spectral spreading to neighbouring frequencies and leakages to higher frequencies. Both can be reduced by using other types of window functions such as Hamming or Hanning, which have smooth cut offs.

3. Discrete Fourier Transform (DFT) is used to calculate the Fourier Transform in a computer. This is done by taking the windowed portion of the signal and construct a periodic signal from it. The result is a sampled Fourier Transform with frequency stop $f_0 = 1/T_0$, where T_0 is the window function width.