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Lecture 6

Systems & Laplace Transform

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In this lecture, I will introduce the idea of a system to which one applies signals. Almost any physical setup can take on a "system" view. Engineers model the system using mathematics. The main goal of system analysis is to be able predict its behaviour under different conditions.

One of the most useful mathematical tools to analyse and thus, predict, systems is the Laplace Transform. This lecture will also introduce the theory of Laplace Transform and show how it may be used to model systems as transfer functions.

Up to now, we have been focusing on the processing of electrical signals. In five short lectures, we have covered quite a lot of ground. It is therefore time to review what you have learned so far. Here are the TEN key teachings of what we have covered up to now:

1.Signals in time-domain and frequency-domain views – This is fundamental to signal processing, Depending on what you want to do with the signal, processing in one of the two domain will proof beneficial. A good example is shown earlier when a sinewave is corrupt by large noise signal. In time-domain, it looks a mess. In frequency-domain, the energy is spread over the entire spectrum and therefore the sinewave is not "masked" by the noise.

2.Any signal can be represented by weighted sum of sinusoids – This is the essence of Fourier transform, and it is how we convert from one domain to another.

3.Sinusoid as sine, cosine or exponential functions – Sinusoids form the "building blocks" of signals in frequency domain. If you project a sinewave of one frequency onto another sinewave of a different frequency, no matter how close they are, the projection is zero. This implies that the two sinewaves are "orthogonal" and they have nothing in common. This is also why sinusoids form good building blocks.

4.Fourier Transform – converts a time-limited signal with finite energy from time-domain to frequency-domain.

$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

10 things you have learned about signals (2)

6. You must sample a signal at a sampling frequency f_s which is at least twice that of maximum signal frequency f_{max}: $f_s \geq 2 \cdot f_{max}$.

7. When sampling signals at f_s, the spectrum of the original signal is repeated at EVERY multiple of sampling frequency, i.e $\pm nf_s$, n = 1, 2, 3...

8. If you sample a signal which has a frequency component higher than fs/2, aliasing occurs (which results in spectral folding).

9. When you extract a portion of a signal, you effectively multiply the signal with a rectangular window, which results in spreading of energy to neigbouring frequency components. This is known as "leakage".

10. You can reduce this leakage by multiplying your signal with a special window function which has smooth instead of shape edges.

5. **Periodic signal uses Fourier series in frequency domain** – The fundamental frequency f0 = 1/T0, the period of the signal, and all other components are called harmonics, and they are integral multiples of f0.

6. **Sampling theorem** – One must sample at fs samples per second, which is at least TWICE that of the maximum frequency of the signal fmax: fs ≥ 2*fmax.

7. **Spectrum of a sample signal** – When you sample a signal, the spectrum of the continuous time signal get repeated indefinitely at multiple of fs, i.e. at $\pm$nfs, where n is all integers except 0: $\pm$1, $\pm$2 ....

8. **Sampling a signal too slowly corrupts it through aliasing** – If you use a sampling frequency fs which is lower than 2*fmax, aliasing, or spectral folding occurs and this will corrupt the signal in a way that you cannot go back to continuous time without error.

9. **Rectangular windows** – When extracting a portion of a signal to analyse, you are effectively multiplying the signal with a rectangular window. This results in leakages – signal energy leaked to its neighbouring frequency components.

10. **Better to use window functions with smooth edges** – Leakages can be reduced significantly by using other they of windowing functions, such as Hamming and Hanning windows.

What are Systems?

- Systems are used to **process signals** to **modify** or **extract information**
- Physical system – characterized by their **input-output relationships**
- E.g. electrical systems are characterized by voltage-current relationships for components and the **laws of interconnections** (i.e. Kirchhoff's laws)
- From this, we derive a **mathematical model** of the system
- "**Black box**" model of a system:
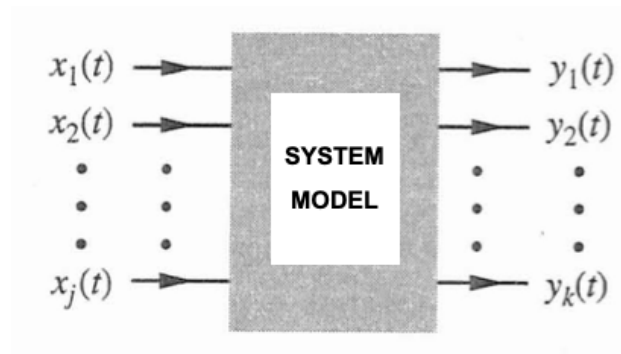
L1.6

Here is a general view of a SYSTEM. It processes signals from the input $x_j(t)$ and produces signals $y_k(t)$ at the output.

What we are attempting to do in this course module is to learn how to characterize and model the input-to-output relationship. For example, we have already learn to calculate the relationship between output voltage and input voltage in an operational amplifier from your Year 1 Electronics 1 module.

Generally, we use mathematics to model the system behaviour, and produce some form of equations relating $y_k(t)$ to $x_j(t)$.

Since we don't really care what is exactly inside the system beyond this input-output relationship, we call this a "Black box" model of the system.

Linear Systems (1)

- A **linear system** exhibits the **additivity** property:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad\qquad x_1 + x_2 \longrightarrow y_1 + y_2$$

- It also must satisfy the **homogeneity** or **scaling** property:

$$x \longrightarrow y \qquad\qquad kx \longrightarrow ky$$

- These can be combined into the property of **superposition**:

$$x_1 \longrightarrow y_1 \quad x_2 \longrightarrow y_2 \qquad\qquad k_1 x_1 + k_2 x_2 \longrightarrow k_1 y_1 + k_2 y_2$$

- A non-linear system is one that is NOT linear (i.e. does not obey the principle of superposition)

L1.7-1

One of the most important property of any system is linearity. A linear system exhibits two important properties: 1) additive: if x1 leads to y1, x2 leads to y2, then x1+x2 leads to y1+y2; 2) scaling: if x leads to y, kx leads to ky.

These two properties can be combined to form the general form of superposition, a principle that we have already covered extensively last year.

Many physical systems are NOT inherently linear. For example, we have already considered that our ears are sensitive to sound volume in a logarithmic manner. An incandescent light bulb produce light output as a quadratic function (i.e. square) of the input voltage.

However, we can usually approximate a non-linear system as linear over a range of signal, particularly if the range is small. Therefore we often perform the so-called "small signal analysis", restricting the signal to perturbation around a certain operating point.

We will examine this in Lab 3 in more details tomorrow.

Linear Systems (2)

- Consider the following simple RC circuit:
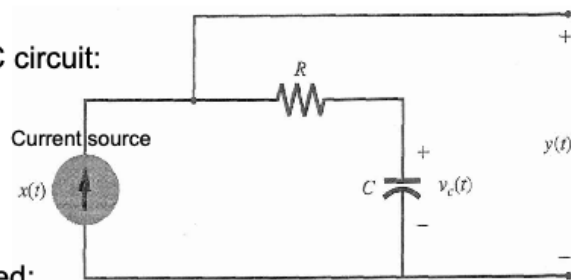- Output *y(t)* relates to *x(t)* by:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

- The second term can be expanded:

$$y(t) = Rx(t) + \frac{1}{C}\int_{-\infty}^{0} x(\tau)d\tau + \frac{1}{C}\int_{0}^{t} x(\tau)d\tau$$

$$y(t) = v_C(0) + Rx(t) + \frac{1}{C}\int_{0}^{t} x(\tau)d\tau$$

- This is a **single-input, single-output** (SISO) system. In general, a system can be multiple-input, multiple-output (**MIMO**).

L1.6 (p100)

Let us take this example from last year – one of a simple RC circuit consisting of a resistor R and a capacitor C. The output is the voltage y(t). The input is the current x(t) from the current source.

Remember that the voltage across the resistor is governed by Ohm's law = R x(t) .

The voltage across the capacitor is 1/C x integral of the current x(t) from τ= -∞ to τ (i.e. now).

If we separate the integral term into two parts, one from τ = -∞ to τ= 0, then τ=0 to τ=t. We get the final equation, which is effective a differential/integral equation where vc(0) is the voltage across the capacitor at time 0 (i.e. the initial condition).

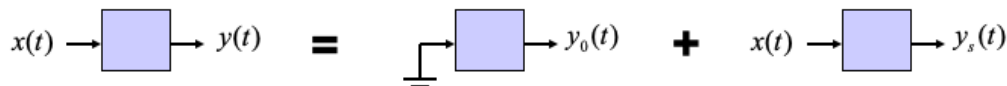This system has one input x(t) and one output y(t), and it is called a single-input, single-output (SISO) system.

You can also have a MIMO system (e.g. you can buy a good wireless access point which uses many antenna and produces multiple wifi signals, and it is known as a MIMO system on the product).

Linear Systems (3)

◆ A system's output for $t \geq 0$ is result of 2 independent causes:
  1. Initial conditions when $t = 0$ (**zero-input response**)
  2. Input $x(t)$ for $t \geq 0$ (**zero-state response**)

◆ Decomposition property:

Total response = zero-input response + zero-state response

$$y(t) = \underbrace{v_C(0)}_{\text{zero-input response}} + \underbrace{Rx(t) + \frac{1}{C}\int_0^t x(\tau)\,d\tau}_{\text{zero-state response}} \qquad t \geq 0$$

$x(t) \longrightarrow \boxed{} \longrightarrow y(t) \quad = \quad \text{---}\boxed{} \longrightarrow y_0(t) \quad + \quad x(t) \longrightarrow \boxed{} \longrightarrow y_s(t)$
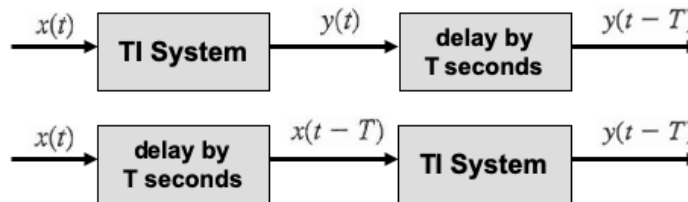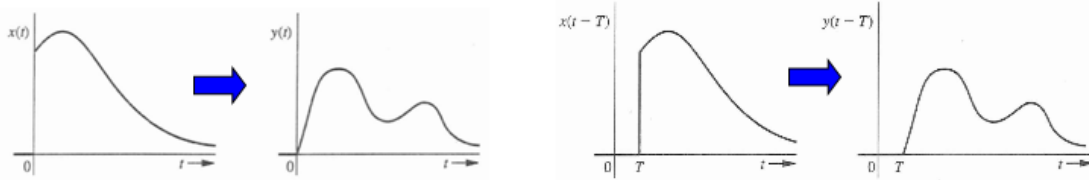
L1.7-1 p102

Now it is important to appreciate that given a system, the output response is made up of two parts:

1. The initial condition, which is also called the zero-input response. This is the system behaviour before any input is applied (as if the input is grounded).

2. The zero-state response. This is the the system behaviour of the system to the input assuming that the internal state (such as the capacitor voltage) are all initially zero.

Time-Invariant Systems

- **Time-invariant system** is one whose parameters do not change with time:

$x(t)$ → **TI System** → $y(t)$ → **delay by T seconds** → $y(t - T)$

$x(t)$ → **delay by T seconds** → $x(t - T)$ → **TI System** → $y(t - T)$

- Linear time-invariant (**LTI**) systems – main concern for this course. (Lathi: LTIC = LTI continuous, LTID = LTI discrete)

L1.7-2 p106

Another important classification of any systems is time-invariant vs time-variant.

A time-invariant system means that the characteristic is NOT change (invariant) over time. It is fixed and no dependent on when you use the system, today, tomorrow or next year.

In this module, we only consider systems that are LINEAR, and TIME-INVARIANT, and call this LTI system for short.

Other classification of systems

- ◆ Systems may be classified into these catagories:
  1. **Linear or non-linear;**
  2. **Time-invariant or time-variant;**
  3. Instantaneous (memoryless) or dynamic (with memory) systems
  4. Causal or non-causal systems
  5. Continuous-time or discrete-time systems
  6. Analogue or digital systems
  7. Invertible or noninvertible systems
  8. Stable or unstable systems

L1.7

There are other classification of systems. You should be familiar with them already. A system could be instantaneous – for example a circuit using only resistors, or dynamic with storage, for example a circuit with capacitor which stores voltages.

Another important one is causal and non-causal systems. A causal system is one that is "idle" for t < 0, and only starts responding (turned ON) at t ≥ 0. We mostly consider causal system in this module.

We have done continuous vs discrete time systems, analogue vs digital systems.

Invertible and noninvertible systems – don't worry too much about this for now. Finally stable vs unstable systems – also don't worry about this for now.
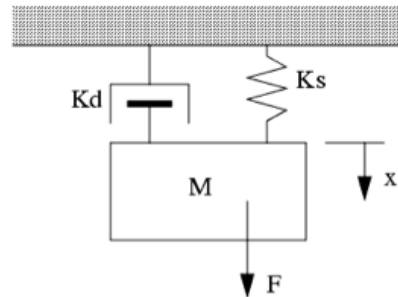
System modelling using ODEs

- ◆ Many systems in electrical and mechanical engineering where input *x(t)* and output loop current *y(t)* are related by **ordinary differential equations (ODEs)**
- ◆ For example:

DE1.3 L8 S11
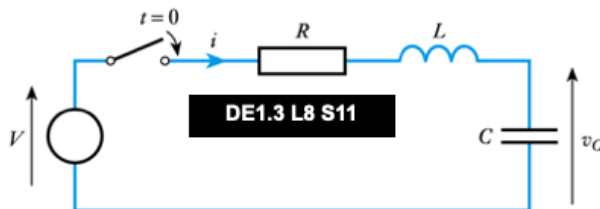
$$v_L(t) + v_R(t) + v_C(t) = V$$

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V$$

$$M\ \ddot{x}(t) + K_d\dot{x}(t) + K_s x(t) = F(t)$$

L1.8

You are familiar with modeling systems with differential equations. The circuit shown here was taken from DE1.3 Lecture 8, slide 11. Assuming that all voltages and currents were 0 for t<0. At t=0, the switch closes. We are interested in find out v_C(t) as a function of time.

You can easily write an equation as shown by summing the voltage around the loop (Kirkoff's voltage law – voltage around a loop in a circuit sums to zero). This provides us with a differential equation, which can be solved for v_C(t).

Similar, consider a mechanical system with a mass M, hanging from the ceiling with a damper with damping coefficient Kd and a spring with a Young's coefficient Ks. If you apply a force F(t) the mass, what is x(t)?

Summing all the forces together in the vertical direction, we get the differential equation shown. The gravitation force is proportional to d^2x/dt^2. The force of the damper if proportional to dx/dt. The force on the spring is proportional to x(t) itself.

Although modeling systems as differential equation works, solving ODE is a bit tedious. Laplace transform is a method to solve ODEs without pain!

$$y(t) = h(t) * x(t)$$

time-domain Analysis using ODEs or Impulse response h(t)

frequency-domain analysis H(ω)
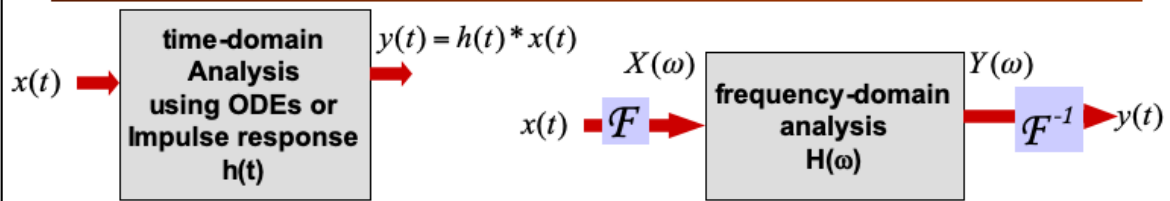
- Analyse system using differential equations or using the system's **impulse response** h(t) (later lecture)
- Analyse system behaviour in time-domain via solving differential equations can be tedious.
- Could use impulse response and convolution (later topic), but could be expensive.
- Using Fourier transforms and frequency response to analyse (and predict behaviour of) a system has limitations.
- Frequency response is only useful in predicting **steady-state behaviour** of a system, not transient behaviour.
- Alternative – use Laplace transform to transform both system and signals to the complex Laplace variable, the s-domain.

Before we consider Laplace transform theory, let us put everything in the context of signals being applied to systems.

If we take a time-domain view of signals and systems, we have the top left diagram. The input x(t) is a function of time (i.e. a waveform you see on a scope), and the system is modeled as ODEs. Alternatively you may also model the time-domain system through its response to an impulse at the input. We will be covering impulse response in a later lecture. The system response to an impulse is known as "impulse response" and is usually represented as h(t). In time-domain analysis, you get y(t) either by solving the ODEs or you could derive y(t) from x(t) and h(t) through an operation known as "convolution". This is again something that will be covered later in this module.

However, if you operate in the frequency domain (from now on, I will drop the hyphen), we take the Fourier transform of the input signal: x(t) ➔ X(ω). We then model the system with its frequency response H(ω). The output (in the frequency domain) Y(ω) is given by   Y(ω) = X(ω) x H(ω), a simple multiplication.

In other words, the frequency response H(ω) is a model of how the system passes (or suppresses) different frequency components in the signal X(ω). This is exactly the process whereby you adjust your model phone playing music to emphasize low frequencies (bass) to get stronger beats in disco music, or emphasize higher frequencies (treble) to gain more clarity in classical music.

Laplace Transform (1)

◆ Laplace Transform is a method that converts differential equations in time-domain into algebraic equations in complex Laplace variable s-domain.

◆ Definition of Laplace Transform $\mathcal{L}$ is:

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

◆ This version of Laplace transform, known as one-sided or unilateral Laplace Transform, is valid only if x(t) = 0 for all t ≤ 0. In other words, x(t) is casual.

◆ Once transformed to the s-domain, analysis and prediction of the system because easy if we know the system's characteristic H(s), which is also called the transfer function (more later)

$X(s)$ $Y(s) = H(s)X(s)$

$x(t) \rightarrow \mathcal{L} \rightarrow$ **s-domain analysis H(s)** $\rightarrow \mathcal{L}^{-1} \rightarrow y(t)$
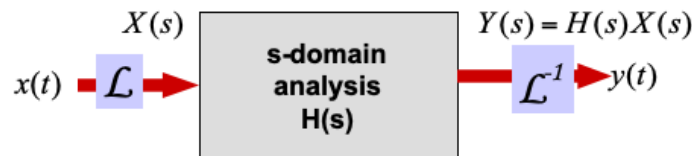
L4.1 p345

Laplace Transform is in someway similar to Fourier Transform. However it is more general, and arguably more powerful.

It converts differential equations in the time domain into algebraic equations in another domain with a complex Laplace variable s. Let us call this the s-domain.

The mathematical definition of the general Laplace Transform (also called bilateral Laplace Transform) is:

$$\mathcal{L}[x(t)] = X(s) = \int_{-\infty}^\infty x(t)e^{-st}\,dt$$

For this course, we assume that the signal and the system are both causal, i.e. x(t) = 0 for all t < 0. Therefore we get the equation shown in the slide, where the limits of integration is from 0 and NOT -∞.

Similar to Fourier domains, we can transform input signal x(t) to the Laplace or s-domain as X(s), and we can model the system in the s-domain using its response H(s). This is also called the Transfer Function.   If you known X(s) and H(s), then the output in the s-domain Y(s) = H(s) X(s) – very similar to the Fourier analysis we did before.

We will consider the relationship (similarity) between Fourier transform and Laplace transform later. For now, you can regard Fourier transform is a special case of Laplace transform. So Laplace is more general.

Laplace Transform (2)

- Laplace Transform obeys laws of linearity:

$$\mathcal{L}\left[\beta_1 x_1(t) + \beta_2 x_2(t)\right] = \beta_1 \mathcal{L}[x_1(t)] + \beta_2 \mathcal{L}[x_2(t)]$$

- The Laplace transform of an impulse function:

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}dt = 1 \qquad \text{for all } s$$

$$\boxed{L[\delta(t)] \quad \Leftrightarrow \quad 1}$$

- The Laplace transform of a unit step function:

$$L[u(t)] = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \qquad \text{Re } s > 0$$

$$\boxed{L[u(t)] \quad \Leftrightarrow \quad \frac{1}{s}}$$

L4.1 p346

Before we go any further, let us consider the Laplace transforms of interesting signals and functions.

First, you must remember that Laplace transform, just like Fourier, obeys the law of linearity – it is a linear tranform.

Now let us consider the Laplace transform of an impulse $\delta(t)$. This simple integration shows that:

This is similar to the case

$$\mathcal{L}[\delta(t)] \quad \Leftrightarrow \quad 1$$

of Fourier transform shown in Lecture 3, slide 7.

The Laplace transform of a unit step signal u(t) is 1/s. Again you can derive this through simple integration. Remember that e^{-st} ➔ 0 when t ➔∞.

Laplace Transform (3)

♦ Laplace Transform of $e^{at}u(t)$:

$$\mathcal{L}[e^{at}u(t)] = \int_0^\infty e^{at}e^{-st}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

$$\mathcal{L}[e^{at}u(t)] \quad \Leftrightarrow \quad \frac{1}{s-a}$$

♦ Laplace Transform of $\cos\omega_0 t\, u(t)$:

$$\mathcal{L}[\cos\omega_0 t\, u(t)] = \frac{1}{2}\mathcal{L}[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)]$$

$$= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] = \frac{s}{s^2+\omega_0^2}$$

$$\mathcal{L}[\cos\omega_0 t u(t)] \quad \Leftrightarrow \quad \frac{s}{s^2+\omega_0^2}$$

L4.1 p347

Now consider Laplace transform of a causal exponential signal e^{at} u(t). (Note that multiplying e^{at} by u(t) makes the signal causal because u(t) chops off everything where t < 0.

Again simple integration yields the result you see.

From this, we can also derive the Laplace transform for a causal cosine signal at frequency ω_0.

Laplace Transform (4)

♦ Laplace Transform of a differentiator $\dot{x}(t) = \dfrac{dx(t)}{dt}$:

$$\mathcal{L}[\frac{dx(t)}{dt}] = \int_{t=0}^{\infty} \frac{dx(t)}{dt} e^{-st} \, dt$$

♦ It can be shown (using integration by parts) that this result in:

$$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$$

♦ If x(0) = 0 (i.e. zero initial condition), then $\mathcal{L}[\dot{x}(t)] = sX(s)$

♦ Therefore, differentiation in the time domain is multiplication by s in the s-domain:

$$\frac{d}{dt} \longleftrightarrow s$$

We can also derive the Laplace transform for a function. For example, what is the LT of a differentiation function d/dt?

As shown here, the result is also pretty simple. x(0) is the initial value of x at t = 0.

If x(0) = 0, i.e. zero initial condition, then L(dx(t)/dt) = s X(s). This is a very important result.

Laplace Transform (5)

- Laplace Transform of an integrator $\int_{\tau=0}^{t} x(\tau)d\tau$:

  Let $g(t) = \int_{\tau=0}^{t} x(\tau)d\tau$

  then $x(t) = \dfrac{dg(t)}{dt}$, and $g(0) = 0$

- From last slide

  $$\mathcal{L}[x(t)] = \mathcal{L}[\dot{g}(t)] = sG(s) - g(0) = sG(s)$$

- Therefore

  $$\mathcal{L}[g(t)] = \frac{1}{s}X(s)$$

- Therefore, integration in the time domain is multiplication by 1/s in the s-domain:

  $$\int_{t=0}^{t} \longleftrightarrow s^{-1}$$

Similarly, we can compute the Laplace transform of the integration function. This is slightly more complicated.

We first express the integration of x(t) as g(t): $g(t) = \int_{\tau=0}^{t} x(\tau)d\tau$

This leads to: $x(t) = \dfrac{dg(t)}{dt}$, and $g(0) = 0$

If we now take Laplace transform on both sides, we get:

$$\mathcal{L}[x(t)] = \mathcal{L}[\dot{g}(t)] = sG(s) - g(0) = sG(s)$$

Therefore LT of an integrator is the same as multiplying the input X(s) by 1/s in the s-domain.

Now we are ready to generalize. Assuming zero initial condition, L[dx/dt] = sX(s), it follows that L[d^2x/dt^2] is s^2X(s) ….. L[d^kx/dt^k] is s^k X(s).

So let us take our mechanical system previously considered in Slide 10. The second-order differential equation:

$$M\ \ddot{x}(t) + K_d \dot{x}(t) + K_s x(t) = F(t)$$

Can be converted to the Laplace s-domain (zero initial condition) as:

$$Ms^2 X(s) + K_d sX(s) + K_s X(s) = F(s)$$

Re-arrange this a bit, and express this as OUTPUT/INPUT in the s-domain, we get:

$$H(s) = \frac{X(s)}{F(x)} = \frac{1}{Ms^2 + K_d s + K_s}$$

This is a very important results. H(s) is known as Transfer function, and it characterizes the system in the s-domain as a 2nd order polynomial function in the complex Laplace variable s. This is an algebraic equation. Since Y(s) = H(s) X(s), a simple multiplication, we can predict the output by simple algebraic calculations. No more fiddling with differential equations!