SAMPLE PAPER (2011)

Time allowed: 2:00 hours

There are 15 questions on this paper.
This paper is accompanied by four tables of formulae.

ANSWER ALL QUESTIONS.
1. Explain briefly what is a linear time-invariant (LTI) system.
   (Lecture 1)

2. For each of the following cases, specify if the signal is even or odd:
   i) \( x(t) = \sin \alpha \)
   ii) \( x(t) = j \cos \alpha \).
   (Lecture 1)

3. A continuous-time signal \( x(t) \) is shown in Figure 1.1. Sketch the signals
   i) \( x(t)[u(t) - u(t - 2)] \)
   ii) \( x(t) \delta(t - 2) \).
   (Lecture 1)

4. Consider the LR circuit shown in Figure 1.2. Find the relationship between the input \( v_s(t) \) and the output \( V_l(t) \) in the form of:
   i) a differential equation;
   ii) a transfer function.
   (Lectures 2 & 7)

5. The unit impulse response of an LTI system is \( h(t) = e^{-t} u(t) \). Use the convolution table, find the system’s zero-state response \( y(t) \) if the input \( x(t) = e^{-2t} u(t) \).
   (Lectures 4/5, Tutorial 3 Q3b)
6. Find and sketch $c(t) = f_1(t) * f_2(t)$ for the pairs of functions as shown in Figure 1.3.

![Figure 1.3](image1)

(Lectures 5, Tutorial 3 Q6c)

7. Find the pole and zero locations for a system with the transfer function

$$H(s) = \frac{s^2 - 2s + 50}{s^2 + 2s + 50}.$$  

(Lectures 8, Tutorial 5 Q9)

8. The Fourier transform of the triangular pulse $f(t)$ shown in Fig. 1.4 (a) is given to be:

$$F(\omega) = \frac{1}{\omega^2} (e^{j\omega} - j\omega e^{j\omega} - 1)$$

Use this information and the time-shifting and time-scaling properties, find the Fourier transforms of the signal $f_1(t)$ shown in Fig. 1.4 (b).

![Figure 1.4](image2)

(Lectures 10, Tutorial 6 Q5)

9. Using the z-transform pairs given in the z-transform table, find the inverse z-transform of

$$X[z] = \frac{z(z - 4)}{z^2 - 5z + 6}.$$  

(Lectures 14/15, Tutorial 8 Q6a)
10. Fig. 1.5 shows Fourier spectra of the signal $f_1(t)$. Determine the Nyquist sampling rates for the signals $f_1(t)$ and $f_2(t)$.

![Fourier spectra](image)

*Figure 1.5*

(Lectures 12, Tutorial 7 Q2a)

11. Find the unit impulse response of the LTI system specified by the equation

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 3y(t) = \frac{dx}{dt} + 5x(t).$$

(Lectures 2 & 3, Tutorial 2 Q5)

12. A line charge is located along the $x$ axis with a charge density $f(x)$. In other words, the charge over an interval $\Delta \tau$ located at $\tau = n\Delta \tau$ is $f(n\Delta \tau)\Delta \tau$. It is also known from Coulomb’s law that the electrical field $E(r)$ at a distance $r$ from a charge $q$ is given by:

$$E(r) = \frac{q}{4\pi \varepsilon r^2}, \quad \text{where } \varepsilon \text{ is a constant.}$$

Show that electric field $E(x)$ produced by this line charge at a point $x$ is given by

$$E(x) = f(x) * h(x), \quad \text{where } h(x) = \frac{q}{4\pi \varepsilon x}.$$

(Lectures 4 & 5, Tutorial 3 Q8)

13. A discrete LTI system specified by the following difference equation:

$$y[n+1] - 0.8y[n] = x[n+1]$$

a) Find its transfer function in the z-domain.

b) Find the frequency response of this discrete time system.

(Lectures 16)

[THE END]
Table of formulae for E2.5 Signals and Linear Systems  
(For use during examination only.)

### Convolution Table

<table>
<thead>
<tr>
<th>No.</th>
<th>$x_1(t)$</th>
<th>$x_2(t)$</th>
<th>$x_1(t) * x_2(t) = x_2(t) * x_1(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x(t)$</td>
<td>$\delta(t - T)$</td>
<td>$x(t - T)$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{t_1}u(t)$</td>
<td>$u(t)$</td>
<td>$1 - e^{t_1} - \frac{-\lambda}{\lambda} u(t)$</td>
</tr>
<tr>
<td>3</td>
<td>$u(t)$</td>
<td>$u(t)$</td>
<td>$tu(t)$</td>
</tr>
<tr>
<td>4</td>
<td>$e^{\lambda_1}u(t)$</td>
<td>$e^{\lambda_2}u(t)$</td>
<td>$\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} u(t) \quad \lambda_1 \neq \lambda_2$</td>
</tr>
<tr>
<td>5</td>
<td>$e^{t_1}u(t)$</td>
<td>$e^{t_2}u(t)$</td>
<td>$te^{t_1}u(t)$</td>
</tr>
<tr>
<td>6</td>
<td>$te^{t_1}u(t)$</td>
<td>$e^{t_2}u(t)$</td>
<td>$\frac{1}{2}t^2e^{t_1}u(t)$</td>
</tr>
<tr>
<td>7</td>
<td>$t^Nu(t)$</td>
<td>$e^{t_1}u(t)$</td>
<td>$\frac{N!e^{t_1}}{\lambda^{N+1}} u(t) - \sum_{k=0}^{N} \frac{N! t^{N-k}}{\lambda^{k+1}(N-k)!} u(t)$</td>
</tr>
<tr>
<td>8</td>
<td>$t^Mu(t)$</td>
<td>$t^Nu(t)$</td>
<td>$\frac{M!N!}{(M + N + 1)!} t^{M+N+1}u(t)$</td>
</tr>
<tr>
<td>9</td>
<td>$te^{t_1}u(t)$</td>
<td>$e^{t_2}u(t)$</td>
<td>$\frac{e^{t_1} - e^{t_2} + (\lambda_1 - \lambda_2)te^{t_1}}{(\lambda_1 - \lambda_2)^2} u(t)$</td>
</tr>
<tr>
<td>10</td>
<td>$t^Me^{t_1}u(t)$</td>
<td>$t^Ne^{t_2}u(t)$</td>
<td>$\frac{M!N!}{(N + M + 1)!} t^{M+N+1}e^{t_1}u(t)$</td>
</tr>
<tr>
<td>11</td>
<td>$t^Me^{t_1}u(t)$</td>
<td>$t^Ne^{t_2}u(t)$</td>
<td>$\sum_{k=0}^{M} \frac{(-1)^k M!(N+k)! t^{M-k}e^{t_1}}{k!(M-k)!((\lambda_1 - \lambda_2)^{N+k+1})} u(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+ \sum_{k=0}^{N} \frac{(-1)^k N!(M+k)! t^{N-k}e^{t_1}}{k!(N-k)!((\lambda_2 - \lambda_1)^{M+k+1})} u(t)$</td>
</tr>
<tr>
<td>12</td>
<td>$e^{-\alpha t} \cos(\beta t + \theta)u(t)$</td>
<td>$e^{t_2}u(t)$</td>
<td>$\frac{\cos(\theta - \phi)e^{t_1} - e^{-\alpha t}\cos(\beta t + \theta - \phi)}{\sqrt{(\alpha + \lambda)^2 + \beta^2}} u(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\phi = \tan^{-1}[\beta/(\alpha + \lambda)]$</td>
</tr>
<tr>
<td>13</td>
<td>$e^{t_1}u(t)$</td>
<td>$e^{t_2}u(-t)$</td>
<td>$\frac{e^{t_1}u(t) + e^{t_2}u(-t)}{\lambda_2 - \lambda_1} \quad \text{Re } \lambda_2 &gt; \text{Re } \lambda_1$</td>
</tr>
<tr>
<td>14</td>
<td>$e^{t_1}u(-t)$</td>
<td>$e^{t_2}u(-t)$</td>
<td>$\frac{e^{t_1} - e^{t_2}}{\lambda_2 - \lambda_1} u(-t)$</td>
</tr>
</tbody>
</table>
### Laplace Transform Table

<table>
<thead>
<tr>
<th>No.</th>
<th>$x(t)$</th>
<th>$X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>3</td>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>4</td>
<td>$t^n u(t)$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>5</td>
<td>$e^{\lambda t} u(t)$</td>
<td>$\frac{1}{s - \lambda}$</td>
</tr>
<tr>
<td>6</td>
<td>$te^{\lambda t} u(t)$</td>
<td>$\frac{1}{(s - \lambda)^2}$</td>
</tr>
<tr>
<td>7</td>
<td>$t^n e^{\lambda t} u(t)$</td>
<td>$\frac{n!}{(s - \lambda)^{n+1}}$</td>
</tr>
<tr>
<td>8a</td>
<td>$\cos bt u(t)$</td>
<td>$\frac{s}{s^2 + b^2}$</td>
</tr>
<tr>
<td>8b</td>
<td>$\sin bt u(t)$</td>
<td>$\frac{b}{s^2 + b^2}$</td>
</tr>
<tr>
<td>9a</td>
<td>$e^{-at} \cos bt u(t)$</td>
<td>$\frac{s + a}{(s + a)^2 + b^2}$</td>
</tr>
<tr>
<td>9b</td>
<td>$e^{-at} \sin bt u(t)$</td>
<td>$\frac{b}{(s + a)^2 + b^2}$</td>
</tr>
<tr>
<td>10a</td>
<td>$re^{-at} \cos (bt + \theta) u(t)$</td>
<td>$\frac{(r \cos \theta) s + (a \cos \theta - b \sin \theta)}{s^2 + 2as + (a^2 + b^2)}$</td>
</tr>
<tr>
<td>10b</td>
<td>$re^{-at} \cos (bt + \theta) u(t)$</td>
<td>$\frac{0.5re^{j\theta}}{s + a - jb} + \frac{0.5re^{-j\theta}}{s + a + jb}$</td>
</tr>
<tr>
<td>10c</td>
<td>$re^{-at} \cos (bt + \theta) u(t)$</td>
<td>$\frac{As + B}{s^2 + 2as + c}$</td>
</tr>
<tr>
<td>10d</td>
<td>$e^{-at} \left[ A \cos bt + \frac{B - Aa}{b} \sin bt \right] u(t)$</td>
<td>$\frac{As + B}{s^2 + 2as + c}$</td>
</tr>
</tbody>
</table>

For $r = \sqrt{\frac{A^2 c + B^2 - 2ABa}{c - a^2}}$ and $
\theta = \tan^{-1} \left( \frac{Aa - B}{A\sqrt{c - a^2}} \right)$, $b = \sqrt{c - a^2}$. 

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### Fourier Transform Table

<table>
<thead>
<tr>
<th>No.</th>
<th>$x(t)$</th>
<th>$X(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e^{-at}u(t)$</td>
<td>$\frac{1}{a + j\omega}$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{at}u(-t)$</td>
<td>$\frac{1}{a - j\omega}$</td>
</tr>
<tr>
<td>3</td>
<td>$e^{-</td>
<td>t</td>
</tr>
<tr>
<td>4</td>
<td>$te^{-at}u(t)$</td>
<td>$\frac{1}{(a + j\omega)^2}$</td>
</tr>
<tr>
<td>5</td>
<td>$i^n e^{-at}u(t)$</td>
<td>$\frac{n!}{(a + j\omega)^{n+1}}$</td>
</tr>
<tr>
<td>6</td>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>$2\pi \delta(\omega)$</td>
</tr>
<tr>
<td>8</td>
<td>$e^{j\omega t}$</td>
<td>$2\pi \delta(\omega - \omega_0)$</td>
</tr>
<tr>
<td>9</td>
<td>$\cos \omega_0 t$</td>
<td>$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$</td>
</tr>
<tr>
<td>10</td>
<td>$\sin \omega_0 t$</td>
<td>$j\pi [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$</td>
</tr>
<tr>
<td>11</td>
<td>$u(t)$</td>
<td>$\pi \delta(\omega) + \frac{1}{j\omega}$</td>
</tr>
<tr>
<td>12</td>
<td>$\text{sgn } t$</td>
<td>$\frac{2}{j\omega}$</td>
</tr>
<tr>
<td>13</td>
<td>$\cos \omega_0 t \ u(t)$</td>
<td>$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>14</td>
<td>$\sin \omega_0 t \ u(t)$</td>
<td>$\frac{\pi}{2j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$</td>
</tr>
<tr>
<td>15</td>
<td>$e^{-at} \sin \omega_0 t \ u(t)$</td>
<td>$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$</td>
</tr>
<tr>
<td>16</td>
<td>$e^{-at} \cos \omega_0 t \ u(t)$</td>
<td>$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$</td>
</tr>
<tr>
<td>17</td>
<td>$\text{rect} \left( \frac{t}{\tau} \right)$</td>
<td>$\tau \text{sinc} \left( \frac{\omega \tau}{2} \right)$</td>
</tr>
<tr>
<td>18</td>
<td>$\frac{W}{\pi} \text{sinc} (Wt)$</td>
<td>$\text{rect} \left( \frac{\omega}{2W} \right)$</td>
</tr>
<tr>
<td>19</td>
<td>$\Delta \left( \frac{t}{\tau} \right)$</td>
<td>$\tau \text{sinc}^2 \left( \frac{\omega \tau}{4} \right)$</td>
</tr>
<tr>
<td>20</td>
<td>$\frac{W}{2\pi} \text{sinc}^2 \left( \frac{Wt}{2} \right)$</td>
<td>$\Delta \left( \frac{\omega}{2W} \right)$</td>
</tr>
<tr>
<td>21</td>
<td>$\sum_{n=-\infty}^{\infty} \delta(t - nT)$</td>
<td>$\omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$</td>
</tr>
<tr>
<td>22</td>
<td>$e^{-t^2/2\sigma^2}$</td>
<td>$\sigma \sqrt{2\pi} e^{-\sigma^2 \omega^2/2}$</td>
</tr>
</tbody>
</table>
## $z$-transform Table

<table>
<thead>
<tr>
<th>No.</th>
<th>$x[n]$</th>
<th>$X[z]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta[n - n]$</td>
<td>$z^{-k}$</td>
</tr>
<tr>
<td>2</td>
<td>$u[n]$</td>
<td>$\frac{z}{z - 1}$</td>
</tr>
<tr>
<td>3</td>
<td>$nu[n]$</td>
<td>$\frac{z}{(z - 1)^2}$</td>
</tr>
<tr>
<td>4</td>
<td>$n^2 u[n]$</td>
<td>$\frac{z(z + 1)}{(z - 1)^3}$</td>
</tr>
<tr>
<td>5</td>
<td>$n^3 u[n]$</td>
<td>$\frac{z(z^2 + 4z + 1)}{(z - 1)^4}$</td>
</tr>
<tr>
<td>6</td>
<td>$\gamma^n u[n]$</td>
<td>$\frac{z}{z - \gamma}$</td>
</tr>
<tr>
<td>7</td>
<td>$\gamma^{n-1}u[n - 1]$</td>
<td>$\frac{1}{z - \gamma}$</td>
</tr>
<tr>
<td>8</td>
<td>$n\gamma^n u[n]$</td>
<td>$\frac{\gamma z}{(z - \gamma)^2}$</td>
</tr>
<tr>
<td>10</td>
<td>$\frac{n(n - 1)(n - 2) \cdots (n - m + 1)}{\gamma^m m!} \gamma^m u[n]$</td>
<td>$\frac{z}{(z - \gamma)^{m+1}}$</td>
</tr>
<tr>
<td>11a</td>
<td>$</td>
<td>\gamma</td>
</tr>
<tr>
<td>11b</td>
<td>$</td>
<td>\gamma</td>
</tr>
<tr>
<td>12a</td>
<td>$r</td>
<td>\gamma</td>
</tr>
<tr>
<td>12b</td>
<td>$r</td>
<td>\gamma</td>
</tr>
<tr>
<td>12c</td>
<td>$r</td>
<td>\gamma</td>
</tr>
</tbody>
</table>

$$r = \sqrt{A^2|\gamma|^2 + B^2 - 2AaB \over |\gamma|^2 - a^2}$$

$$\beta = \cos^{-1} \frac{-a}{|\gamma|}$$

$$\theta = \tan^{-1} \frac{Aa - B}{A \sqrt{|\gamma|^2 - a^2}}$$