Imperial College London

Lecture 6

Frequency-domain analysis: Laplace Transform

(Lathi 4.1 – 4.2)

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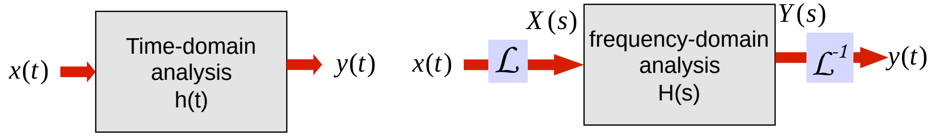
Why Laplace Transform?

- Laplace transform is the **dual** (or complement) of the time-domain analysis.
- In **time-domain** analysis, we break input x(t) into **impulsive component**, and sum the system response to all these components.
- In **frequency-domain** analysis, we break the input x(t) into **exponentials components** of the form e^{st}, where s is the complex frequency: $s = \alpha + j\omega$
- Laplace transform is the tool to map signals and system behaviour from the time-domain into the frequency domain.



Definition of Two-sided Laplace Transform

- For a signal *x(t)*, its **Laplace transform is defined by:**

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

- The signal *x(t)* is said to be the **inverse Laplace transform** of X(s). It can be shown that

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

  where c is a constant chosen to ensure the **convergence** of the first integral.
- Note that this definition is slightly more complex than what you have seen in Dr Jaimouka's 2nd year Control course. This general definition does not assume casuality.
- This general definite is known as **two-sided** (or **bilateral**) Laplace Transform.

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Definition of One-sided Laplace Transform

- For the purpose of the 2nd year curriculum, let us assume that **all signals are causal**. For this the Laplace transform is defined as:

$$X(s) = \mathcal{L}[x(t)] = \int_{0}^{\infty} x(t) e^{-st} dt$$

- This is the same as that defined on the 2nd year Control course, and is known as **one-side** (or **unilateral**) Laplace transform.
- Remember that the Laplace transform is **a linear tranform** (see Jamouka's notes, p15):

$$\mathcal{L}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 \mathcal{L}\{f_1(t)\} + k_2 \mathcal{L}\{f_2(t)\}$$

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A few examples

- Find the Laplace transform of δ(t) and u(t).

$$\mathcal{L}[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = 1 \qquad \text{for all } s$$

$$\mathcal{L}[\delta(t)] \quad \Leftrightarrow \quad 1$$

$$\mathcal{L}[u(t)] = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt$$

$$= -\frac{1}{s} e^{-st} \Big|_0^\infty = \frac{1}{s} \qquad \text{Re } s > 0$$

$$\mathcal{L}[u(t)] \quad \Leftrightarrow \quad \frac{1}{s}$$

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A few examples (2)

- Find the Laplace transform of $e^{at} u(t)$ and $\cos \omega_0 t\, u(t)$.

$$\mathcal{L}[e^{at} u(t)] = \int_0^\infty e^{at} e^{-st} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}$$

$$\mathcal{L}[e^{at} u(t)] \quad \Leftrightarrow \quad \frac{1}{s-a}$$

$$\mathcal{L}[\cos \omega_0 t\, u(t)] = \frac{1}{2} \mathcal{L}[e^{j\omega_0 t} u(t) + e^{-j\omega_0 t} u(t)]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] = \frac{s}{s^2 + \omega_0^2}$$

$$\mathcal{L}[\cos \omega_0 t u(t)] \quad \Leftrightarrow \quad \frac{s}{s^2 + \omega_0^2}$$

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Laplace transform Pairs (1)

- Finding inverse Laplace transform requires integration in the complex plane – beyond scope of this course.
- So, use a Laplace transform table (analogous to the convolution table).

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$

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Laplace transform Pairs (2)

No.	$x(t)$	$X(s)$
5	$e^{\lambda t} u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t} u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t} u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at} \cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at} \sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$

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Laplace transform Pairs (3)

No.	$x(t)$	$X(s)$
10a	$re^{-at}\cos(bt+\theta)u(t)$	$\dfrac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)u(t)$	$\dfrac{0.5re^{j\theta}}{s+a-jb}+\dfrac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)u(t)$	$\dfrac{As+B}{s^2+2as+c}$
	$r=\sqrt{\dfrac{A^2c+B^2-2ABa}{c-a^2}}$	
	$\theta=\tan^{-1}\left(\dfrac{Aa-B}{A\sqrt{c-a^2}}\right)$	
	$b=\sqrt{c-a^2}$	
10d	$e^{-at}\left[A\cos bt+\dfrac{B-Aa}{b}\sin bt\right]u(t)$	$\dfrac{As+B}{s^2+2as+c}$
	$b=\sqrt{c-a^2}$	

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Examples of Inverse Laplace Transform (1)

- Finding inverse Laplace transform of $\dfrac{7s-6}{s^2-s-6}$. (use partial fraction)

$$X(s)=\frac{7s-6}{(s+2)(s-3)}=\frac{k_1}{s+2}+\frac{k_2}{s-3}$$

- To find k_1 which corresponds to the term (s+2), cover up (s+2) in X(s), and substitute s = -2 (i.e. s+2=0) in the remaining expression:

$$k_1=\left.\frac{7s-6}{(s+2)(s-3)}\right|_{s=-2}=\frac{-14-6}{-2-3}=4$$

- Similarly for k_2:

$$k_2=\left.\frac{7s-6}{(s+2)(s-3)}\right|_{s=3}=\frac{21-6}{3+2}=3$$

- Therefore

$$X(s)=\frac{7s-6}{(s+2)(s-3)}=\frac{4}{s+2}+\frac{3}{s-3}$$

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Examples of Inverse Laplace Transform (2)

- Easy to make mistake with partial fraction.
- Method to check correctness of:

$$X(s)=\frac{7s-6}{(s+2)(s-3)}=\frac{4}{s+2}+\frac{3}{s-3}$$

- Substitute s = 0 into the equation (could use other values, but this is most convenient):

$$X(0)=\frac{-6}{(+2)(-3)}=1=\frac{4}{2}+\frac{3}{-3}$$

- Therefore, using Pair 5 from table:

$$x(t)=\mathcal{L}^{-1}\left(\frac{4}{s+2}+\frac{3}{s-3}\right)=(4e^{-2t}+3e^{3t})u(t)$$

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Examples of Inverse Laplace Transform (3)

- Finding the inverse Laplace transform of $\dfrac{2s^2-5}{(s+1)(s+2)}$.
- The partial fraction of this expression is less straight forward. If the power of numerator polynomial (M) is the same as that of denominator polynomial (N), we need to add the coefficient of the highest power in the numerator to the normal partial fraction form:

$$X(s)=2+\frac{k_1}{s+1}+\frac{k_2}{s+2}$$

- Solve for k_1 and k_2 via "covering":

$$k_1=\left.\frac{2s^2+5}{(s+1)(s+2)}\right|_{s=-1}=\frac{2+5}{-1+2}=7$$

$$k_2=\left.\frac{2s^2+5}{(s+1)(s+2)}\right|_{s=-2}=\frac{8+5}{-2+1}=-13$$

- Therefore $X(s)=2+\dfrac{7}{s+1}-\dfrac{13}{s+2}$
- Using pairs 1 & 5:

$$x(t)=2\delta(t)+(7e^{-t}-13e^{-2t})u(t)$$

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Time Shifting Property of the Laplace transform

- **Time Shifting** property:

$$x(t) \Longleftrightarrow X(s)$$
$$x(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad \text{for } t_0 \geq 0$$

- Delaying $x(t)$ by t_0 (i.e. time shifting) amounts to multiplying its transform $X(s)$ by e^{-st_0}.
- Remember that $x(t)$ starts at $t = 0$, and $x(t - t_0)$ starts at $t = t_0$.
- Therefore, the more accurate statement of the time shifting property is:

$$x(t)u(t) \Longleftrightarrow X(s)$$
$$x(t-t_0)u(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad t_0 \geq 0$$

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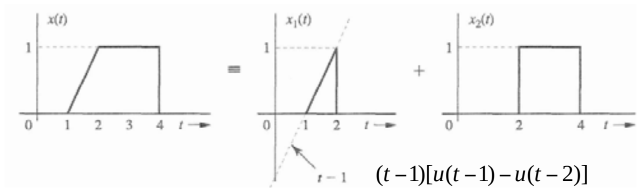
Application of Time Shifting

- Find the Laplace transform of $x(t)$ as shown:

$[u(t-2)-u(t-4)]$



$(t-1)[u(t-1)-u(t-2)]$

$$x(t) = (t-1)[u(t-1)-u(t-2)] + [u(t-2)-u(t-4)]$$
$$= (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4)$$
$$= (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$$

$u(t) \Longleftrightarrow \frac{1}{s}$

$tu(t) \Longleftrightarrow 1/s^2$

Time shift

$(t-2)u(t-2) \Longleftrightarrow \frac{1}{s^2}e^{-2s}$

$(t-1)u(t-1) \Longleftrightarrow \frac{1}{s^2}e^{-s}$

$u(t-4) \Longleftrightarrow \frac{1}{s}e^{-4s}$

$$X(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

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Frequency Shifting Property

- **Frequency Shifting** property:

$$x(t) \Longleftrightarrow X(s)$$
$$x(t)e^{s_0 t} \Longleftrightarrow X(s-s_0)$$

- Frequency shifting the transform $X(s)$ by s_0 amounts to multiplying its time signal by $e^{s_0 t}$.
- Observe symmetry (or duality) between frequency-shift and time-shift properties.

$$x(t) \Longleftrightarrow X(s)$$
$$x(t-t_0) \Longleftrightarrow X(s)e^{-st_0} \qquad \text{for } t_0 \geq 0$$

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Application of Frequency Shifting

- Given $\cos bt\, u(t) \Longleftrightarrow \frac{s}{s^2+b^2}$, show that $e^{-at}\cos bt\, u(t) \Longleftrightarrow \frac{s+a}{(s+a)^2+b^2}$.
- Apply frequency-shifting property with frequency shift $s_0 = -a$.
- Replace s with (s+a) means frequency shift by -a. This yields the RHS of the equation. By frequency-shifting property, we need to multiply the LHS by e^{-at}.

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Time-Differentiation Property

- **Time-differentiation** property:

$$x(t) \Longleftrightarrow X(s)$$
$$\frac{dx}{dt} \Longleftrightarrow sX(s) - x(0^-)$$

- Repeated application of this property yields:

$$\frac{d^2x}{dt^2} \Longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-)$$
$$\frac{d^nx}{dt^n} \Longleftrightarrow s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$
$$= s^nX(s) - \sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$$

where $x^{(r)}(0^-)$ is d^rx/dt^r at $t = 0^-$.

- Frequency-differentiation property:

$$x(t) \Longleftrightarrow X(s)$$
$$tx(t) \Longleftrightarrow -\frac{d}{ds}X(s)$$

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Proof of Time-Differentiation Property

$$\mathcal{L}\left[\frac{dx}{dt}\right] = \int_{0^-}^{\infty} \frac{dx}{dt} e^{-st}\, dt$$

- Integration by parts gives:

$$\mathcal{L}\left[\frac{dx}{dt}\right] = x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st}\, dt$$

- For the Laplace integral to converge, it is necessary that

$$x(t)e^{-st} \to 0 \text{ as } t \to \infty$$

- Therefore we get:

$$\mathcal{L}\left[\frac{dx}{dt}\right] = -x(0^-) + sX(s)$$

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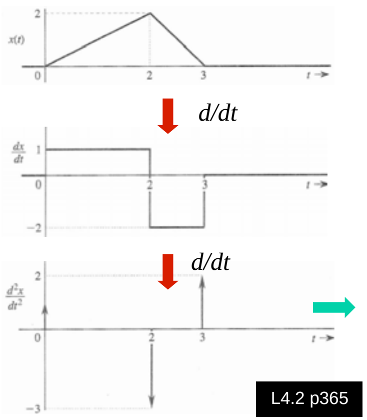
Application of Time-Differentiation

- Find the Laplace transform of the signal x(t) using time differentiation and time-shifting properties.



$$X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$$

↑

$$s^2X(s) - 0 - 0 = 1 - 3e^{-2s} + 2e^{-3s}$$

↑ $x(0^-) = \dot{x}(0^-) = 0$, and $\delta(t) \Longleftrightarrow 1$

$$\frac{d^2x}{dt^2} \Longleftrightarrow s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

↑ Time-differentiation & time shifting properties

$$\frac{d^2x}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$
$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = \mathcal{L}[\delta(t) - 3\delta(t-2) + 2\delta(t-3)]$$

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Time-Integration Property

- **Time-integration** property:

$$x(t) \Longleftrightarrow X(s)$$
$$\int_{0^-}^{t} x(\tau)\, d\tau \Longleftrightarrow \frac{X(s)}{s}$$

- The dual property of time-integration is the frequency-integration property:

$$x(t) \Longleftrightarrow X(s)$$
$$\frac{x(t)}{t} \Longleftrightarrow \int_{s}^{\infty} X(z)\, dz$$

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Scaling Property

- **Scaling property**:

$$x(t) \Longleftrightarrow X(s)$$
$$x(at) \Longleftrightarrow \frac{1}{a}X\left(\frac{s}{a}\right) \quad \text{for } a > 0$$

- Time compression of a signal by a factor a causes expansion of its Laplace transform in s-scale by the same factor.

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Time-Convolution & Frequency-Convolution Properties

- **Time-convolution** property:

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$
$$x_1(t) * x_2(t) \Longleftrightarrow X_1(s)X_2(s)$$

- Convolution in time domain is equivalent to multiplication in s (frequency) domain.
- **Frequency-convolution** property:

$$x_1(t) \Longleftrightarrow X_1(s) \quad \text{and} \quad x_2(t) \Longleftrightarrow X_2(s)$$
$$x_1(t)x_2(t) \Longleftrightarrow \frac{1}{2\pi j}[X_1(s) * X_2(s)]$$

- **Convolution** in s (frequency) domain is equivalent to **multiplication** in time domain.

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Application of the convolution Properties

- Use the time-convolution property of the Laplace transform to determine

$$c(t) = e^{at}u(t) * e^{bt}u(t).$$

- Since $e^{at}u(t) \Leftrightarrow \dfrac{1}{(s-a)}$ $\qquad e^{bt}u(t) \Leftrightarrow \dfrac{1}{(s-b)}$

- Therefore $e^{at}u(t) * e^{bt}u(t) \Leftrightarrow \dfrac{1}{(s-a)(s-b)}$

$$C(s) = \frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]$$

- Perform inverse Laplace transform gives:

$$c(t) = \frac{1}{a-b}(e^{at} - e^{bt})u(t)$$

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Relationship with time-domain analysis

- If $h(t)$ is the impulse response of a LTI system, then we have see in lectures 4 & 5 that the system response $y(t)$ to an input $x(t)$ is $x(t)*h(t)$.
- Assuming causality, and that $h(t) \Leftrightarrow H(s)$ and $x(t) \Leftrightarrow X(s)$ then

$$Y(s) = X(s)H(s)$$

- The response $y(t)$ is the zero-state response of the LTI system to the input $x(t)$. It follows that the transfer function $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[\text{zero-state response}]}{\mathcal{L}[\text{input}]}$$

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Summary of Laplace Transform Properties (1)

Operation	$x(t)$	$X(s)$
Addition	$x_1(t)+x_2(t)$	$X_1(s)+X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\frac{dx}{dt}$	$sX(s)-x(0^-)$
	$\frac{d^2x}{dt^2}$	$s^2X(s)-sx(0^-)-\dot{x}(0^-)$
	$\frac{d^3x}{dt^3}$	$s^3X(s)-s^2x(0^-)-s\dot{x}(0^-)-\ddot{x}(0^-)$
	$\frac{d^nx}{dt^n}$	$s^nX(s)-\sum_{k=1}^{n}s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)$
	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)+\frac{1}{s}\int_{-\infty}^{0^-}x(t)\,dt$

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Summary of Laplace Transform Properties (2)

Operation	$x(t)$	$X(s)$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0}$ $\quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\frac{dX(s)}{ds}$
Frequency integration	$\frac{x(t)}{t}$	$\int_s^{\infty}X(z)\,dz$
Scaling	$x(at),\ a\geq 0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Time convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi j}X_1(s)*X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s)$ $\quad (n>m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s)$ $\quad$ [poles of $sX(s)$ in LHP]

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Relating this lecture to other courses

- You have done Laplace transform in maths and in control courses. This lecture is mostly a revision, plus emphasis on the convolution – multiplication properties for the two domains.
- Many of the properties are deliberately stated without proofs. It is more important on this course to understand the actual interpretations of Laplace transform (and more importantly the duality of time and frequency domains) than the mathematic proofs.