Lecture 7
More on Laplace Transform
(Lathi 4.3 – 4.4)

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Initial & Final Value Theorems

How to find the initial and final values of a function \( x(t) \) if we know its Laplace Transform \( X(s) \)? (\( t \to 0^+ \), and \( t \to \infty \))

**Initial Value Theorem**

Conditions:
- Laplace transforms of \( x(t) \) and \( \frac{dx}{dt} \) exist.
- \( X(s) \) numerator power (M) is less than denominator power (N), i.e. \( M<N \).

\[ \lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s) \]

**Final Value Theorem**

Conditions:
- Laplace transforms of \( x(t) \) and \( \frac{dx}{dt} \) exist.
- \( sX(s) \) poles are all on the Left Plane or origin.

\[ \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \]

Example

Find the initial and final values of \( y(t) \) if \( Y(s) \) is given by:

\[ Y(s) = \frac{10(2s + 3)}{s(s^2 + 2s + 5)} \]

Initial value: \( y(0^+) = \lim_{s \to \infty} sY(s) \)

\[ = \lim_{s \to \infty} \frac{10(2s + 3)}{s^2 + 2s + 5} = 0 \]

Final value: \( y(\infty) = \lim_{s \to 0} sY(s) \)

\[ = \lim_{s \to 0} \frac{10(2s + 3)}{s^2 + 2s + 5} = 6 \]

Laplace Transform for Solving Differential Equations

Remember the time-differentiation property of Laplace Transform:

\[ \frac{d^2y}{dt^2} \iff s^2Y(s) \]

Exploit this to solve differential equation as algebraic equations:

\[ x(t) \xrightarrow{\text{time-domain analysis}} \text{solve differential equations} \]

\[ y(t) \]

\[ x(t) \xrightarrow{\text{frequency-domain analysis}} \text{solve algebraic equations} \]

\[ X(s) \xrightarrow{\mathcal{L}^{-1}} y(t) \]
Example (1)

- Solve the following second-order linear differential equation:
  \[ \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + x(t) \]
- Given that \( y(0^-) = 2 \), \( y'(0^-) = 1 \) and input \( x(t) = e^{-t}u(t) \).

### Time Domain

\[
\begin{align*}
\frac{dy}{dt} &= sY(s) - y(0^-) = sY(s) - 2 \\
\frac{d^2y}{dt^2} &= s^2Y(s) - sy(0^-) - y'(0^-) = s^2Y(s) - 2s - 1 \\
x(t) &= e^{-t}u(t) \\
\frac{dx}{dt} &= sX(s) - x(0^-) = s - 0 = s
\end{align*}
\]

### Laplace (Frequency) Domain

\[
\begin{align*}
Y(s) &= \frac{2s + 11}{s^2 + 5s + 6} \\
X(s) &= \frac{s + 1}{s + 4}
\end{align*}
\]

\[
y(t) = \left( \frac{7e^{-2t} - 5e^{-3t}}{s + 2} \right) u(t) + \left( -\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-t} \right) u(t)
\]

Zero-input & Zero-state Responses

- Let’s think about where the terms come from:
  \( (s^2 + 5s + 6)Y(s) - (2s + 11) = \frac{s + 1}{s + 4} \)
  \( \text{Initial condition term} \)
  \( \text{Input term} \)

\[
Y(s) = \frac{2s + 11}{s^2 + 5s + 6} + \frac{s + 1}{(s + 4)(s^2 + 5s + 6)}
\]

\[
y(t) = \left( 7e^{-2t} - 5e^{-3t} \right) u(t) + \left( -\frac{1}{2}e^{-2t} + 2e^{-3t} - \frac{3}{2}e^{-t} \right) u(t)
\]

Example (2)

### Time Domain

\[
\begin{align*}
\frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 6y(t) &= \frac{dx}{dt} + x(t) \\
&= \frac{s^2Y(s) - 2s - 1}{s + 4} + \frac{5[sY(s) - 2] + 6Y(s)}{s + 4} \\
&= \frac{s^2 + 5s + 6}[(s + 4)(s + 2) + 20s + 45]
\end{align*}
\]

### Laplace (Frequency) Domain

\[
\begin{align*}
Y(s) &= \frac{2s^2 + 20s + 45}{(s + 2)(s + 3)(s + 4)} \\
y(t) &= \left( \frac{13/2}{s + 3} - \frac{3/2}{s + 4} \right) e^{-t}u(t)
\end{align*}
\]

Laplace Tranform and Transfer Function

- Let’s express input \( x(t) \) as a linear combination of exponentials \( e^{\alpha t} \):
  \( x(t) = \sum_{i=1}^{k} X(s_i) e^{\alpha_i t} \)

- \( H(s) \) can be regarded as the system’s response to each of the exponential components, in such a way that the output \( y(t) \) is:
  \( y(t) = \sum_{i=1}^{k} X(s_i)H(s_i)e^{\alpha_i t} \)

- Therefore, we get
  \( Y(s) = H(s)X(s) \)

\[
\begin{align*}
x(t) &\xrightarrow{L} X(s) \\
\text{System response to } X(s)e^{at} \text{ is } H(s)X(s) \\
x(t) &\xrightarrow{L^{-1}} y(t)
\end{align*}
\]

\[
\begin{align*}
Y(s) &= H(s)X(s) \\
\text{Transfer Function} \quad H(s)
\end{align*}
\]
**Transfer Function Examples**

- **Delay by T sec**
  \[ x(t) \rightarrow \text{Delay by T sec} \rightarrow y(t) = x(t-T) \]
  
  Shifting Property

- **Differentiator**
  \[ x(t) \rightarrow \text{Differentiator} \rightarrow y(t) = \frac{dx}{dt} \]
  
  Differentiation Property

- **Integrator**
  \[ x(t) \rightarrow \text{Integrator} \rightarrow y(t) = \int x(t) \, dt \]
  
  Integration Property

**Initial conditions in systems (1)**

- In circuits, initial conditions may not be zero. For example, capacitors may be charged; inductors may have an initial current.
- How should these be represented in the Laplace (frequency) domain?
- Consider a capacitor C with an initial voltage \( v(0^-) \):
  \[ i(t) = C \frac{dv}{dt} \]
- Now take Laplace transform on both sides:
  \[ V(s) = \frac{1}{s} I(s) + \frac{v(0^-)}{s} \]
  
  Voltage across charged capacitor
  
  Voltage across capacitor with no charge
  
  Effect of the initial charge = voltage source

**Initial conditions in systems (2)**

- Similarly, consider an inductor L with an initial current \( i(0^-) \):
- Consider a capacitor C with an initial voltage \( v(0^-) \):
- Now take Laplace transform on both sides:
  \[ V(s) = L[sI(s) - i(0^-)] \]
  
  Voltage across inductor
  
  Voltage across inductor with no initial current
  
  Effect of the initial current = voltage source

**Solving Transient Behaviour in circuits – Example 1(1)**

- The switch in the circuit here is in closed position for a long time before \( t=0 \), when it is opened instantaneously. Find the current \( y_1(t) \) and \( y_2(t) \) for \( t > 0 \).
- First determine the initial condition at \( t = 0^- \).
Example 1 (2)

- From this we can rewrite as in matrix form:
- We need to solve for \( Y_1(s) \) and \( Y_2(s) \).
- We do this by applying Cramer’s rule, which is:

\[
\text{Given } A z = c, \quad \text{where } A \text{ is a square matrix, } z \text{ and } c \text{ are column vectors, the vector } z \text{ can be solved by: }

z = \frac{\text{det}(A_i)}{\text{det}(A)}
\]

where \( A_i \) is the matrix \( A \) with its \( i \)th column replaced by column vector \( c \).

\[
\begin{bmatrix}
\frac{1}{s+1} & -\frac{1}{s+1} \\
\frac{1}{s+2} & \frac{1}{s+2}
\end{bmatrix}
\begin{bmatrix}
Y_1(s) \\
Y_2(s)
\end{bmatrix}
= \begin{bmatrix}
\frac{4}{s} \\
\frac{1}{s}
\end{bmatrix}
\]

\[
\text{We readily obtain:}
\]

\[
\text{and therefore:}
\]

\[
Y_1(s) = \frac{24(s+2)}{s^2+7s+12} = \frac{1}{10s} (s^2+7s+12)
\]

\[
Y_2(s) = \frac{24(s+2)}{s^2+7s+12} = \frac{48}{s+4}
\]

Example 1 (3)

- Inverse Laplace gives us:
- Similarly we obtain:
- Therefore:

\[
Y_1(t) = (-24e^{-7t} + 48e^{-4t})u(t)
\]

\[
Y_2(t) = (16e^{-3t} - 12e^{-4t})u(t)
\]

Solving Transient Behaviour in circuits – Example 2(1)

- Find the transfer function \( H(s) \) relating the output \( v_o(t) \) to the input voltage \( v_i(t) \) for the Sallen and Key filter shown below. Assume that initial condition is zero.
- Step 1: Form equivalent circuit
- Step 2: Pick “variables” – nodal voltages at a and b

Solving Transient Behaviour in circuits – Example 2(2)

- Step 3: Sum current at node a
  \[
  \frac{V_a(s) - V_i(s)}{R_1} + \frac{V_i(s) - V_b(s)}{R_2} + \frac{[V_a(s) - K V_b(s)] C_1 s}{R_1} = 0
  \]
  \[
  \frac{1}{R_1} + \frac{1}{R_2 + C_1 s}
  \]

- Step 4: Sum current at node b
  \[
  \frac{V_a(s) - V_b(s)}{R_2} + C_2 s V_b(s) = 0
  \]
  \[
  -\frac{1}{R_2} V_b(s) + \left(\frac{1}{R_2} + C_2 s\right) V_b(s) = 0
  \]

- Step 5: Put in matrix form

\[
\begin{bmatrix}
G_1 + G_2 + C_1 s & -(G_2 + K C_1 s) \\
G_2 & -(G_2 + K C_1 s)
\end{bmatrix}
\begin{bmatrix}
V_a(s) \\
V_b(s)
\end{bmatrix}
= \begin{bmatrix}
G_1 V_i(s) \\
0
\end{bmatrix}
\]

\[
G_1 = \frac{1}{R_1}, \quad G_2 = \frac{1}{R_2}, \quad K = \frac{1}{R_b} + \frac{1}{R_a}
\]
Solving Transient Behaviour in circuits – Example 2(3)

Step 6: Apply Cramer’s rule

\[
\frac{V_i(s)}{V_i(s)} = \frac{G_1G_2}{C_1C_2s^2 + [G_1C_2 + G_2C_2 + G_2C_1(1 - K)]s + G_1G_2} = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}
\]

Step 7: Derive H(s)

\[
K = 1 + \frac{R_e}{R_i} \quad \text{and} \quad \omega_0^2 = \frac{G_1G_2}{C_1C_2} = \frac{1}{\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_2C_2}(1 - K)}
\]

\[
2\alpha = \frac{G_1C_2 + G_2C_2 + G_2C_1(1 - K)}{C_1C_2}
\]

\[
V_o(s) = KV_i(s)
\]

\[
H(s) = \frac{V_o(s)}{V_i(s)} = \frac{K\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}
\]

Relating this lecture to other courses

- You have done much of the circuit analysis in your first year, but Laplace transform provides much more elegant method in finding solutions to BOTH transient and steady state condition of circuits.
- You have done Sallen-and-Key filter in your 2nd year analogue circuits course. Here we derive the transfer function from first principle, using only tools you know about.
- The treatment provided in this lecture also enhances what you have been learning in your 2nd year control course.