Imperial College London

Lecture 8

Frequency Response

(Lathi 4.8 – 4.9)

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Frequency Response of a LTI System

- We have seen that LTI system response to $x(t)=e^{st}$ is $H(s)e^{st}$. We represent such input-output pair as:

$$e^{st} \Rightarrow H(s)e^{st}$$

- Instead of using a complex frequency, let us set s = jω, this yields:

$$e^{j\omega t} \Rightarrow H(j\omega)e^{j\omega t}$$

$$\cos \omega t = \mathrm{Re}(e^{j\omega t}) \Rightarrow \mathrm{Re}[H(j\omega)e^{j\omega t}]$$

- It is often better to express H(jω) in polar form:

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

Frequency Response

- Therefore

$$\cos \omega t \Rightarrow \underline{|H(j\omega)|}\ \cos[\omega t + \underline{\angle H(j\omega)}]$$

Amplitude Response

Phase Response

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Frequency Response Example (1)

- Find the frequency response of a system with transfer function:

$$H(s) = \frac{s+0.1}{s+5}$$

- Then find the system response $y(t)$ for input $x(t)=\cos 2t$ and $x(t)=\cos(10t-50°)$

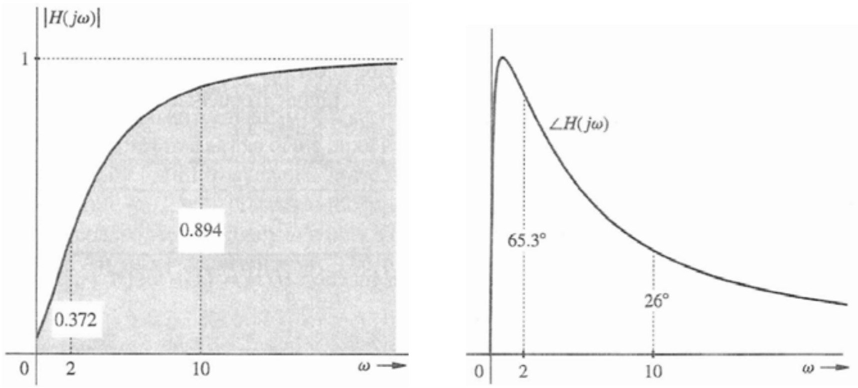
- Substitute $s=j\omega$

$$H(j\omega) = \frac{j\omega+0.1}{j\omega+5}$$

$$|H(j\omega)| = \frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}} \quad \text{and} \quad \angle H(j\omega) = \Phi(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

Frequency Response Example (2)

$$|H(j\omega)| = \frac{\sqrt{\omega^2+0.01}}{\sqrt{\omega^2+25}} \qquad \Phi(j\omega) = \angle H(j\omega) = \tan^{-1}\left(\frac{\omega}{0.1}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$



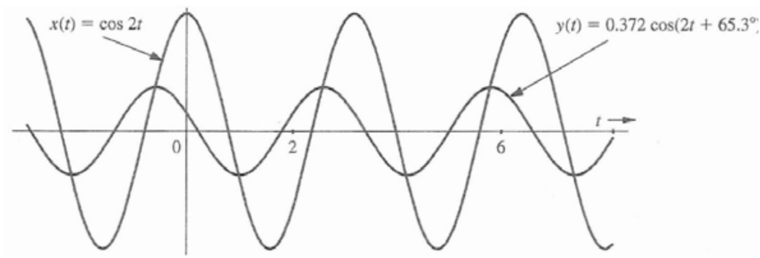
for input $x(t)=\cos 2t$ and $x(t)=\cos(10t-50°)$

Frequency Response Example (3)

- For input *x(t)=cos2t*, we have:

$$|H(j2)| = \frac{\sqrt{2^2+0.01}}{\sqrt{2^2+25}} = 0.372 \qquad \Phi(j2) = \tan^{-1}\left(\frac{2}{0.1}\right) - \tan^{-1}\left(\frac{2}{5}\right) = 65.3^o$$

- Therefore $\quad y(t) = 0.372\cos(2t + 65.3^o)$



Frequency Response Example (4)

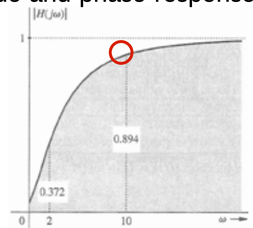
- For input *x(t)=cos(10t-50°)*, we will use the amplitude and phase response curves directly:

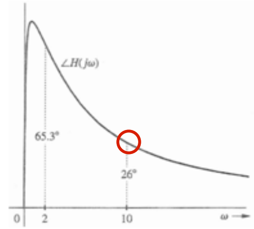
$$|H(j10)| = 0.894$$

$$\Phi(j10) = \angle H(j10) = 26^o$$

- Therefore

$$y(t) = 0.894\cos(10t - 50^o + 26^o) = 0.894\cos(10t + 24^o)$$





Frequency Response of delay of T sec

- H(s) of an ideal T sec delay is:

$$H(s) = e^{-sT} \qquad \text{(Time-shifting property)}$$

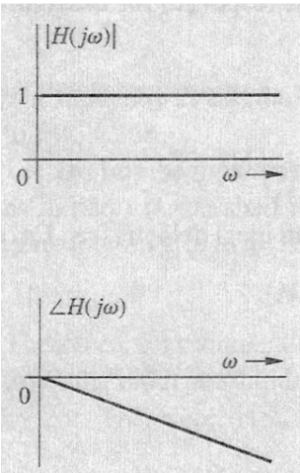
- Therefore

$$|H(j\omega)| = \left|e^{-j\omega T}\right| = 1 \quad \text{and} \quad \Phi(j\omega) = -\omega T$$

- That is, delaying a signal by T has no effect on its amplitude.
- It results in a linear phase shift (with frequency), and a gradient of –T.
- The quantity:

$$-\frac{d\Phi(\omega)}{d\omega} = \tau_g = \text{T}$$

is known as **Group Delay**.



Frequency Response of an ideal differentiator

- H(s) of an ideal differentiator is:

$$H(s) = s \quad \text{and} \quad H(j\omega) = j\omega = \omega e^{j\pi/2}$$

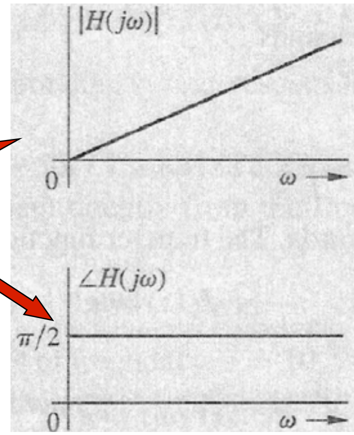
- Therefore

$$|H(j\omega)| = \omega \quad \text{and} \quad \angle H(j\omega) = \frac{\pi}{2}$$

- This agrees with:

$$\frac{d}{dt}(\cos\omega t) = -\omega\sin\omega t = \omega\cos(\omega t + \pi/2)$$

- That's why differentiator is not a nice component to work with – it amplifies high frequency component (i.e. noise!).



Frequency Response of an ideal integrator

- H(s) of an ideal integrator is:

$$H(s)=\frac{1}{s} \quad \text{and} \quad H(j\omega)=\frac{1}{j\omega}=\frac{-j}{\omega}=\frac{1}{\omega}e^{-j\pi/2}$$

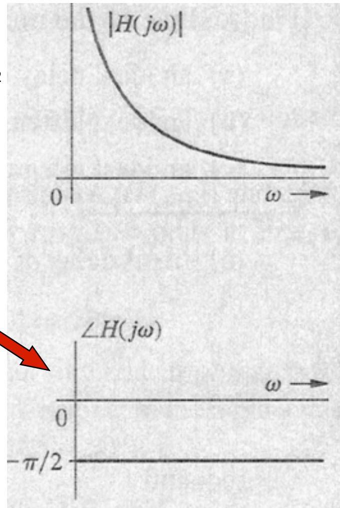
- Therefore

$$|H(j\omega)|=\frac{1}{\omega} \quad \text{and} \quad \angle H(j\omega)=-\frac{\pi}{2}$$

- This agrees with:

$$\int \cos\omega t\, dt=\frac{1}{\omega}\sin\omega t=\frac{1}{\omega}\cos(\omega t-\pi/2)$$

- That's why integrator is a nice component to work with – it suppresses high frequency component (i.e. noise!).



Bode Plot – Sketching frequency response….. without (much) calculation (1)

- Consider a system transfer function:

$$H(s)=\frac{K(s+a_1)(s+a_2)}{s(s+b_1)(s^2+b_2s+b_3)}=\frac{Ka_1a_2}{b_1b_3}\,\frac{\left(\frac{s}{a_1}+1\right)\left(\frac{s}{a_2}+1\right)}{s\left(\frac{s}{b_1}+1\right)\left(\frac{s^2}{b_2}+\frac{b_2}{b_3}s+1\right)}$$

- The **POLES** are roots of the denominator polynomial. In this case, the poles of the system are: $s=0$, $s=-b_1$ and the solutions of the quadratic

$$(s^2+b_2s+b_3)=0$$

which we assume to be complex conjugates values.

- The **ZEROS** are roots of the numerator polynomial. In this case, the zeros of the system are: $s=-a_1$, $s=-a_2$.

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Bode Plot – Sketching frequency response….. without (much) calculation (2)

- Now let s=jω,

$$H(j\omega)=\frac{Ka_1a_2}{b_1b_3}\,\frac{\left|1+\frac{j\omega}{a_1}\right|\left|1+\frac{j\omega}{a_2}\right|}{|j\omega|\left|1+\frac{j\omega}{b_1}\right|\left|1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_2}\right|}$$

- Express this as decibel (i.e. 20 log(…)):

zeros at $-a_1$ and $-a_2$

poles at 0

$$20\log|H(j\omega)|=20\log\frac{Ka_1a_2}{b_1b_3}+20\log\left|1+\frac{j\omega}{a_1}\right|+20\log\left|1+\frac{j\omega}{a_2}\right|-20\log|j\omega|$$

constant term

$$-20\log\left|1+\frac{j\omega}{b_1}\right|-20\log\left|1+\frac{jb_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right|$$

conjugate poles

poles at $-b_1$

- Now amplitude response (in dB) is broken into building block components that are added together.

Building blocks for Bode Plots – amplitude (1)

- Pole term $-20\log|j\omega|$
- Zero term $+20\log|j\omega|$
- Pole term $-20\log\left|1+\frac{j\omega}{a}\right|$

for $\omega \ll a$,

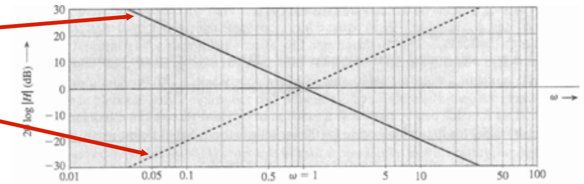
$$-20\log\left|1+\frac{j\omega}{a}\right|\approx -20\log 1=0$$

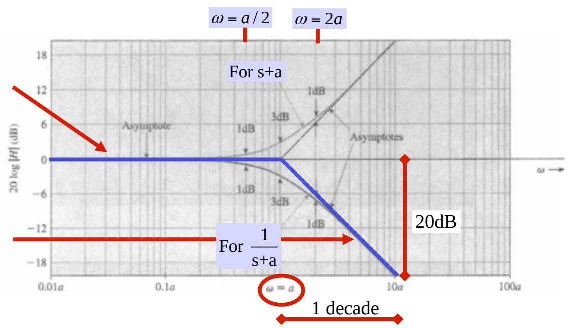
for $\omega \gg a$,

$$-20\log\left|1+\frac{j\omega}{a}\right|\approx -20\log\left(\frac{\omega}{a}\right)$$
$$=-20\log\omega+20\log a$$

at $\omega=a$,

$$-20\log|1+j|=-20\log(\sqrt{2})\approx -3dB$$





Building blocks for Bode Plots – amplitude (2)

- Now consider the quadratic poles: $s^2 + b_2 s + b_3$
- Better to express this as: $s^2 + 2j\varsigma\omega_n s + \omega_n^2$

damping factor
natural frequency

- The log magnitude response is:

$$\text{log amplitude} = -20\log\left|1 + 2j\zeta\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2\right|$$

For $\omega \ll \omega_n$, $\quad$ log amplitude $\approx -20\log 1 = 0$

For $\omega \gg \omega_n$, $\quad$ log amplitude $\approx -20\log\left|\left(-\frac{\omega}{\omega_n}\right)^2\right| = -40\log\left(\frac{\omega}{\omega_n}\right)$

$$= -40\log\omega - 40\log\omega_n$$

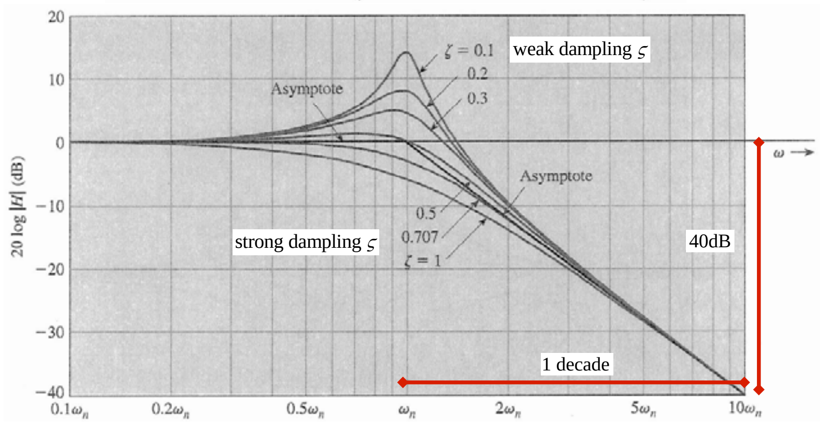
The exact log amplitude is

$$\text{log amplitude} = -20\log\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}$$

Building blocks for Bode Plots – amplitude (3)

- Elsewhere, the exact log amplitude is:

$$\text{log amplitude} = -20\log\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}^{1/2}$$



Bode Plots Example – amplitude (1)

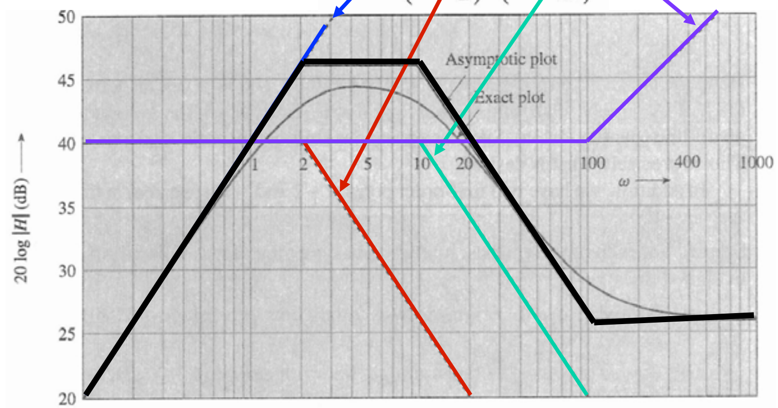
- Consider this transfer function: $H(s) = \dfrac{20s(s+100)}{(s+2)(s+10)}$
- We re-write this as

$$H(s) = \frac{20 \times 100}{2 \times 10}\,\frac{s\left(1 + \frac{s}{100}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)} = 100\frac{s\left(1 + \frac{s}{100}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$

- Step 1: Establish where x-axis crosses the y-axis
  - Since the constant term is 100 = 40dB, x-axis cut the vertical axis at 40.
- Step 2: For each pole and zero term, draw an asymptotic plot.
  - We need to draw straight lines for zeros at origin and ω=100.
  - We need to draw straight line for poles at ω=2 and ω=10.
- Step 3: Add all the asymptotes.
- Step 4: Apply corrections if necessary.

Bode Plots Example – amplitude (2)

$$H(s) = 100\frac{s\left(1 + \frac{s}{100}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$



Building blocks for Bode Plots – Phase (1)

- Now consider phase response for the earlier transfer function:

$$H(j\omega) = \frac{Ka_1a_2}{b_1b_3}\,\frac{\left|1+\frac{j\omega}{a_1}\right|\left|1+\frac{j\omega}{a_2}\right|}{|j\omega|\left|1+\frac{j\omega}{b_1}\right|\left|1+j\frac{b_2\omega}{b_3}+\frac{(j\omega)^2}{b_2}\right|}$$

- Therefore:

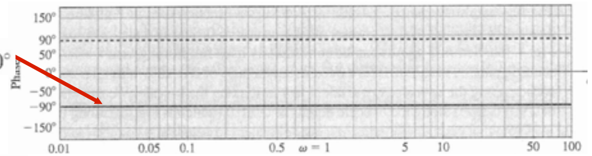
$$\angle H(j\omega) = \angle\left(1+\frac{j\omega}{a_1}\right) + \angle\left(1+\frac{j\omega}{a_2}\right) - \angle j\omega$$
$$-\angle\left(1+\frac{j\omega}{b_1}\right) - \angle\left[1+\frac{jb_2\omega}{b_3}+\frac{(j\omega)^2}{b_3}\right]$$

- Again, we have three type of terms.

Building blocks for Bode Plots – Phase (2)

- Pole term

$$\angle H(j\omega) = -\angle j\omega = -90^\circ$$



- Pole term

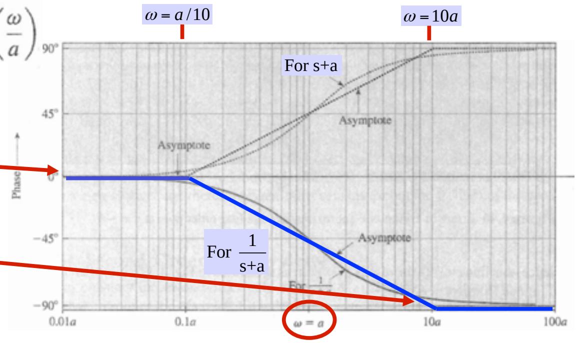
$$\angle H(j\omega) = -\angle\left(1+\frac{j\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

for $\omega \ll a$,

$$-\tan^{-1}\left(\frac{\omega}{a}\right) \approx 0$$

for $\omega \gg a$,

$$-\tan^{-1}\left(\frac{\omega}{a}\right) \approx -90^\circ$$



Building blocks for Bode Plots – Phase (3)

- 2[nd] order poles: $s^2 + 2j\zeta\omega_n s + \omega_n^2$
- Phase response is:

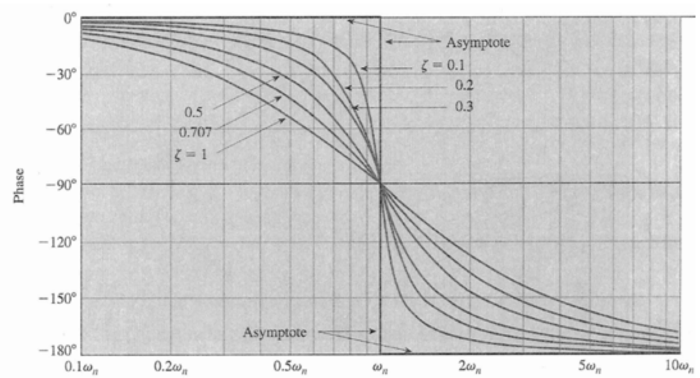
$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

For $\omega \ll \omega_n$,

$$\angle H(j\omega) \approx 0$$

For $\omega \gg \omega_n$,

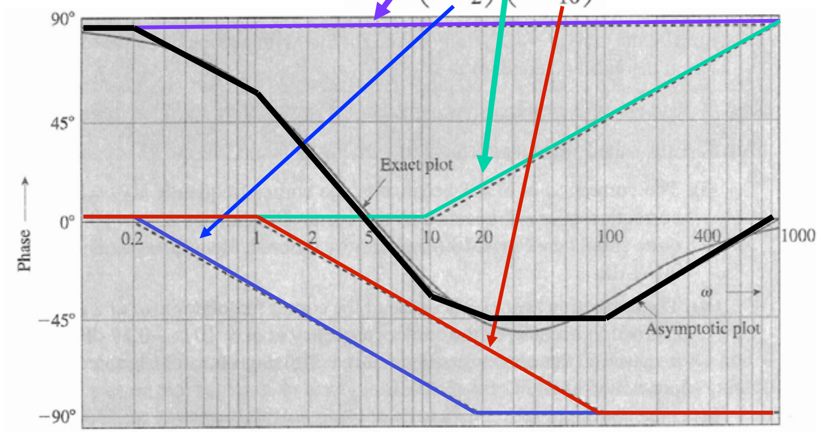
$$\angle H(j\omega) \simeq -180^\circ$$



Bode Plots Example – phase (1)

- Consider this again:

$$H(s) = 100\frac{s\left(1+\frac{s}{100}\right)}{\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$



Relating this lecture to other courses

- You will be applying frequency response in various areas such as filters and 2nd year control. You have also used frequency response in the 2nd year analogue electronics course. Here we explore this as a special case of the general concept of complex frequency, where the real part is zero.
- You have come across Bode plots from 2nd year analogue electronics course. Here we go deeper into where all these rules come from.
- We will apply much of what we done so far in the frequency domain to analyse and design some filters in the next lecture.