A Generalized Form of Tellegen's Theorem

P. PENFIELD, JR., MEMBER, IEEE, ROBERT SPENCE, MEMBER, IEEE, AND S. DUINKER, SENIOR MEMBER, IEEE

Abstract—Among the theorems of circuit theory, Tellegen's theorem is unusual in that it depends solely upon Kirchhoff's laws and the topology of the network. The theorem therefore applies to all electrical networks that obey Kirchhoff's laws, whether they be linear or nonlinear, time-invariant or time-variant, reciprocal or nonreciprocal, hysteretic or nonhysteretic; the excitation is arbitrary, and the initial conditions are also immaterial. When specific assumptions are made concerning the network elements, the excitation, and the initial conditions, Tellegen's theorem reduces to many useful network theorems.

In this paper a generalized form of Tellegen's theorem that allows the efficient derivation of new results is presented. A special form—the "difference form"—of this theorem is shown to be of particular value, and also capable of simple expression in terms of wave variables. The application of the generalized form of Tellegen's theorem is illustrated by an example.

I. Introduction

FROM time to time in a particular field there may be developed a theorem of exceptional value and versatility, which is simple and general, and aids the derivation of known results as well as pointing the way to new ones. In circuit theory, Tellegen's theorem [1]–[3] is of this nature.[1] It states that if $i_1', i_2' \cdots i_b'$ are the branch currents of a b-branch network N', and $v_1'', v_2'' \cdots v_b''$ are the branch voltages of another b-branch network N'', where N' and N'' have a common linear graph but may otherwise be different, then

$$\sum_{\alpha=1}^{b} i_\alpha' v_\alpha'' = 0, \tag{1}$$

where the summation is over all branches (α) of the network. The sign convention adopted for branch voltages and currents is such that, if N' and N'' were identical, the product $i_\alpha'\ v_\alpha''$ would be the instantaneous power supplied to the branch.

Tellegen's theorem is unusual in that only Kirchhoff's laws are invoked in its proof. The theorem therefore applies to all electrical networks that obey these laws, whether they be linear or nonlinear, time-invariant or time-variant, reciprocal or nonreciprocal, passive or active, hysteretic or nonhysteretic. The excitation is arbitrary; it may be sinusoidal, exponential, periodic, transient, or random. The initial conditions are also immaterial.

In this paper we present a generalized form of Tellegen's theorem that we believe to be new. We also present a new "difference form" of the theorem, and show that it is capable of useful and simple expression in terms of wave variables.

II. Tellegen's Theorem

Consider a network having b branches, n_t nodes, and s separate parts. Kirchhoff's current law places $n_t - s$ constraints upon the currents, so that only $b - n_t + s$ currents may be specified independently. All the remaining branch currents may then be found by means of the linear relations

$$i_\alpha = \sum_\beta B_{\beta\alpha} j_\beta, \tag{2}$$

where i_α denotes a general branch current, j_β are the independent currents $b - n_t + s$ in number, $B_{\beta\alpha}$ is the $(b - n_t + s) \times b$ loop matrix of the network, and $\alpha = 1, 2, \cdots b$.

Kirchhoff's voltage law may also be expressed in terms of $B_{\beta\alpha}$. For each arbitrary current there is one closed path within the remainder of the network that does not include any other branch whose current is independently specified. Thus there are $b - n_t + s$ such loops, for each of which Kirchhoff's voltage law may be written as

$$\sum_{\alpha=1}^{b} B_{\beta\alpha} v_\alpha = 0, \tag{3}$$

where the summation is over all branches in the loop.

From Kirchhoff's laws, as expressed by (2) and (3), a simple power theorem can be proved. In fact, because its derivation exactly parallels that of Tellegen's theorem, we consider it briefly here.

Multiplication of (2) by v_α yields

$$i_\alpha v_\alpha = \sum_\beta j_\beta B_{\beta\alpha} v_\alpha. \tag{4}$$

If this is summed over all α (that is, over all branches of the network), then, because of (3), the right-hand side of (4) vanishes so that

$$\sum_{\alpha=1}^{b} i_\alpha v_\alpha = 0. \tag{5}$$

The physical interpretation of (5) is, of course, the conservation of energy within a network. Note that the

Manuscript received June 23, 1969; revised November 6, 1969. This work was supported in part by the Joint Services Electronics Program under Contract DA28-043-AMC-02536(E), and in part by the National Science Foundation, Washington, D. C.

P. Penfield is with the Department of Electrical Engineering and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass. 02139.

R. Spence is with the Department of Electrical Engineering, Imperial College of Science and Technology, London, England.

S. Duinker was with Philip's Zentrallaboratorium, GmbH, Hamburg, Germany. He is now with N. V. Optische Industrie "de Oude Delft," the Netherlands, and the Department of Technical Physics, State University of Groningen, Groningen, the Netherlands.

[1] Recently, Kishi and Kida [4] have given an "edge-port conservation theorem" that has many of the features of Tellegen's theorem.

proof of (5), in drawing exclusively upon Kirchhoff's laws, is valid irrespective of the nature of the circuit elements or of the excitation.

Equation (5) can be extended to networks having two states. By different states of a network we mean the currents and voltages pertinent to different excitations, different element types or values, and/or different initial conditions, *but the same topology and branch numbering.* In other words, the two states of a network may be thought of as the actual states of two different networks that have the same topology. Kirchhoff's laws apply to each state. Thus, for one state of the network, we can write

$$i'_\alpha = \sum_\beta B_{\beta\alpha} j'_\beta \tag{6}$$

and, for the other state,

$$\sum_\alpha B_{\beta\alpha} v''_\alpha = 0, \tag{7}$$

where single and double primes are used to distinguish the two network states. In (6) and (7) the term $B_{\beta\alpha}$ appears unmodified because the two states have the same topology. The same steps that led from (2) and (3) to (5) now lead from (6) and (7) to

$$\sum_\alpha i'_\alpha v''_\alpha = 0. \tag{8}$$

If some branches are, in fact, ports of the network, the products associated with the ports can conveniently be placed on the opposite side of the equality sign to yield

$$\sum_\alpha i'_\alpha v''_\alpha = \sum_p i'_p v''_p, \tag{9}$$

where α and p now denote internal branches, and ports, respectively.[2] This equation may be termed a "quasi-power theorem" [5]. Equation (8) is the theorem originally presented by Tellegen [1], [2], and has since been known, deservedly, as Tellegen's theorem. Valuable though it is, it is a special case of a more general form of the theorem to be derived below.

III. Generalized Form of Tellegen's Theorem

The generalized form of Tellegen's theorem will be expressed in terms of "Kirchhoff operators." The purpose of these operators is to derive, from one set of currents (or voltages) that obeys Kirchhoff's current (or voltage) law, another set of quantities that obeys the law. For example, if the set of currents $\{i_\alpha(t)\}$ obeys Kirchhoff's current law, then so do their time derivatives $\{di_\alpha(t)/dt\}$. Thus, one example of a Kirchhoff current operator is differentiation with respect to time. Another is the Fourier or Laplace transform. Similarly, an operator is called a Kirchhoff voltage operator if, when operating upon a set of voltages that obeys Kirchhoff's voltage law, it generates a set of branch "voltages" that also obeys this law. The term "Kirchhoff operator" will be employed to mean either a Kirchhoff current operator or a Kirchhoff voltage operator, whichever is appropriate in the context. Many Kirchhoff operators, including the examples quoted above, are both current and voltage operators but, as will be shown later, this is not always the case.

Let Λ' be a Kirchhoff current operator whose effect upon the set of branch currents i_α of a b-branch network is the generation of a new set of b branch "currents" $\Lambda' i_\alpha$ that obeys Kirchhoff's current law. Similarly, let Λ'', a Kirchhoff voltage operator, operate upon the set of branch voltages v_α to generate a new set of branch "voltages" $\Lambda'' v_\alpha$ that obeys Kirchhoff's voltage law. For a network with ports it then follows immediately from (9) that

$$\sum_\alpha \Lambda' i_\alpha \Lambda'' v_\alpha = \sum_p \Lambda' i_p \Lambda'' v_p \tag{10}$$

where i_p and v_p are the port currents and voltages, respectively, and the indices α and p are over all the branches and ports of the network. This generalized form of Tellegen's theorem holds for any Kirchhoff operators Λ' and Λ'' and, because it is derived solely from Kirchhoff's laws, is valid for any constitutive laws of the elements, for any form of excitation, and for any initial conditions. Either or both of the Kirchhoff operators may, in fact, consist of a sequence of Kirchhoff operators applied in any order that makes sense.

In the examples quoted earlier in this section, the Kirchhoff operator was applied separately to each of the branch currents or voltages. But in general the operators can be applied to the whole set of currents or voltages. Thus, an example of a topologically dependent Kirchhoff voltage operator is that which selects the differences between the squares of the nodal potentials to form branch "voltages" that obey Kirchhoff's voltage law. This operator, incidentally, is not a Kirchhoff current operator.

Nevertheless, many of the operators used in practice have the property that in generating, say, $\Lambda' i_2$, the other branch currents i_1, i_3, i_4, etc., are ignored. In this event the operator Λ' can also be applied to the independent currents j_β. Thus, if Λ' is a Kirchhoff current operator, the current law can be expressed as

$$\Lambda' i_\alpha = \sum_\beta B_{\beta\alpha} (\Lambda' j_\beta). \tag{11}$$

Substitution of (2) into (11) then yields

$$\Lambda' \Big(\sum_\beta B_{\beta\alpha} j_\beta\Big) = \sum_\beta B_{\beta\alpha} (\Lambda' j_\beta). \tag{12}$$

Because this condition must hold for arbitrary j_β, it follows that Λ' *must be linear.* Since $B_{\beta\alpha}$ is real, it is not necessary that Λ' operate in a linear way on complex numbers. Indeed, complex conjugation is, within our sense of the term, a linear operator.[3] Although linear operators find frequent use in the application of the generalized form of Tellegen's theorem, we emphasize

[2] Use of the convention that $i_p v_p$ is the instantaneous power *entering* a port ensures the absence of a minus sign in (9).

[3] The use of linear operators does not restrict the theorem to linear networks.

that many Kirchhoff operators exist that are not linear.

If Λ' operates on the entire set of currents $\{i_\alpha\}$ rather than on each i_α individually, (11) is meaningless because the effect of Λ' operating on the independent currents j_β has not been defined. However, if Λ' is a Kirchhoff current operator, then the set of "currents" $\{\Lambda' i_\alpha\}$ obeys Kirchhoff's current law, and hence can be written as the matrix product of $B_{\beta\alpha}$ times its own set of independent "currents." That is, a formula like (11) is valid with $\Lambda' j_\beta$ replaced by some suitable quantities.

It can be useful to express Tellegen's theorem in vector-space notation. In a b-branch network, let $\mathcal{V}$ and $\mathcal{I}$ be the sets of all b-dimensional vectors that obey Kirchhoff's voltage and current law, respectively. Then, Tellegen's theorem (8) is a statement that $\mathcal{V}$ and $\mathcal{I}$ are orthogonal subspaces of b-dimensional vector space. That is, any vector in $\mathcal{V}$ is orthogonal to any vector in $\mathcal{I}$. If an operator Λ' maps a b-dimensional vector that obeys Kirchhoff's current law onto the same space $\mathcal{I}$, and an operator Λ'' similarly maps a b-dimensional vector of $\mathcal{V}$ onto $\mathcal{V}$, then the generalized form of Tellegen's theorem (10) results.

In many applications of the generalized form of Tellegen's theorem it is simpler to apply what is called the difference form of the theorem [3]. This form also permits the simple expression of Tellegen's theorem in terms of wave variables. Its derivation is simple: if the roles of Λ' and Λ'' in (10) are interchanged and the result is subtracted from (10), we obtain

$$\sum_\alpha (\Lambda' i_\alpha \Lambda'' v_\alpha - \Lambda'' i_\alpha \Lambda' v_\alpha) = \sum_p (\Lambda' i_p \Lambda'' v_p - \Lambda'' i_p \Lambda' v_p), \qquad (13)$$

which we shall refer to as the difference form of Tellegen's theorem.[4] Clearly, the operators appearing in (13) must be both Kirchhoff current operators *and* Kirchhoff voltage operators.

IV. Wave Variables

Consider some branch of the network having voltage v_α and current i_α. Using any real positive quantity Z_α^n having the dimension of resistance and known as the normalization impedance, we define [6] an incoming wave a_α and an outgoing wave b_α by

$$a_\alpha = \frac{v_\alpha + Z_\alpha^n i_\alpha}{2\sqrt{Z_\alpha^n}} \qquad (14)$$

$$b_\alpha = \frac{v_\alpha - Z_\alpha^n i_\alpha}{2\sqrt{Z_\alpha^n}}. \qquad (15)$$

The waves a_α and b_α are functions of time, although we can define frequency-domain functions by taking their Fourier transforms. In general, wave variables can be defined at each branch and each port, and there is no requirement that all normalization impedances be equal. If the variables i_α and v_α appearing in the generalized form of Tellegen's theorem are expressed, from (14) and (15), in terms of wave variables, the result is somewhat unwieldy. If the same substitutions are made in the *difference* form of the theorem (13), however, we obtain

$$\begin{aligned}\sum_\alpha (\Lambda' i_\alpha \Lambda'' v_\alpha - \Lambda'' i_\alpha \Lambda' v_\alpha) &= 2 \sum_\alpha (\Lambda' a_\alpha \Lambda'' b_\alpha - \Lambda'' a_\alpha \Lambda' b_\alpha) \\ &= \sum_p (\Lambda' i_p \Lambda'' v_p - \Lambda'' i_p \Lambda' v_p) \\ &= 2 \sum_p (\Lambda' a_p \Lambda'' b_p - \Lambda'' a_p \Lambda' b_p). \qquad (16)\end{aligned}$$

Again, the operators Λ' and Λ'' must be both Kirchhoff current operators and Kirchhoff voltage operators. Note that the wave variables appear in the same manner as the currents and voltages, so that many of the results that can be derived from Tellegen's theorem for immittance matrices also hold for scattering matrices. Note also that the contribution for both the ports and the internal branches may be evaluated, at will, in terms of voltage and current variables or wave variables. Thus we might, for example, equate the first and fourth summations in (16), so that a sum of wave-variable terms associated with the ports is equated to a sum of voltage-current terms associated with the internal branches.

V. Example

The generalized form of Tellegen's theorem, and the use of wave variables, can be illustrated by a theorem concerned with the sensitivity of the driving-point impedance of a linear one-port to small variations of its internal, nonreciprocal elements.

Consider a one-port possessing a driving-point impedance Z, and containing linear reciprocal and nonreciprocal elements described by their impedance matrices $[Z_{\alpha\beta}]$, Fig. 1(a). Denote the complex port voltage and current at a frequency ω by V and I, respectively; the same symbols, with appropriate subscripts, are employed for internal variables. Besides the original network, consider another network with the same topology but whose elements are described by a branch-impedance matrix $[\tilde{Z}_{\alpha\beta}]$, which is the transpose of the branch-impedance matrix $[Z_{\alpha\beta}]$ of the corresponding element in the original network, Fig. 1(b). We denote the second network by a superscript tilde (~). Thus,

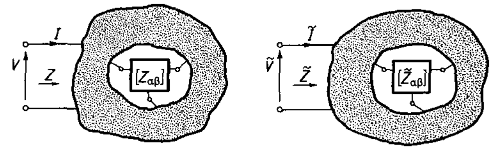
$$\tilde{Z}_{\alpha\beta} = Z_{\beta\alpha}. \qquad (17)$$

Application of the difference form of Tellegen's theorem (13), in which Λ' selects the second network and Λ'' selects small variations in the first network, yields

$$\tilde{I}\,\delta V - \tilde{V}\,\delta I = \sum_\alpha (\tilde{I}_\alpha\,\delta V_\alpha - \tilde{V}_\alpha\,\delta I_\alpha). \qquad (18)$$

Now it can easily be shown that the driving-point impedances of the two networks are identical (i.e., $\tilde{Z} = Z$); a relation which, when combined with (17) and (18),

[4] See also the conclusion of [4].



(a) (b)

Fig. 1. (a) A linear one-port network. (b) Its "adjoint."

yields [3]

$$\delta Z = \sum_{\alpha,\beta} \frac{\tilde{I}_\alpha I_\beta}{\tilde{I}\, I}\, \delta Z_{\alpha\beta}. \tag{19}$$

The second "transpose" network introduced to allow the simple expression or calculation of driving-point sensitivity is often known as the "adjoint" or "inter-reciprocal" [7] of the original network, and has recently enjoyed renewed popularity as the basis of a method of automated circuit design [8].

If preferred, the left-hand side of (18) can be expressed in terms of wave variables to yield

$$\tilde{A} A\ \delta\Gamma = \sum_{\alpha,\beta} \tilde{I}_\alpha I_\beta\ \delta Z_{\alpha\beta} \tag{20}$$

where $\delta\Gamma$ is a small change in the port reflection coefficient, and A and $\tilde{A}$ are the complex amplitudes of the incident wave in the network and its adjoint, respectively.

VI. Conclusions

A generalized form of Tellegen's theorem has been presented. The value of the theorem lies in the fact that it is valid for *all* electrical networks, in the wide available choice of operators and, in the difference form, in its alternative expression in terms of wave variables. An example was presented to illustrate an application of the theorem; many others exist [3].

Acknowledgment

It is a pleasure to acknowledge the helpful communication the authors have had with Prof. C. A. Desoer.

References

[1] B. D. H. Tellegen, "A general network theorem, with applications," *Philips Res. Rept.*, vol. 7, pp. 259–269, August 1952.
[2] B. D. H. Tellegen, "A general network theorem, with applications," *Proc. Inst. Radio Engrs.* (Australia), vol. 14, pp. 265–270, November 1953.
[3] P. Penfield, Jr., R. Spence, and S. Duinker, *Tellegen's Theorem and Electrical Networks.* Cambridge, Mass.: M.I.T. Press, 1970.
[4] G. Kishi and T. Kida, "Edge-port conservation in networks," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-15, pp. 274–276, September 1968.
[5] S. Duinker, "The relationship between various energy distribution theorems," in *Network and Switching Theory*, G. Biorci, Ed. New York: Academic Press, 1968, pp. 1–20.
[6] H. J. Carlin, "The scattering matrix in network theory," *IRE Trans. Circuit Theory*, vol. CT-3, pp. 88–97, June 1956.
[7] J. L. Bordewijk, "Inter-reciprocity applied to electrical networks," *Appl. Sci. Res.*, vol. B6, pp. 1–74, 1956.
[8] S. W. Director and R. A. Rohrer, "Automated network design—The frequency-domain case," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 330–337, August 1969.

Power-Series Equivalence of Some Functional Series With Applications

MARTIN SCHETZEN, MEMBER, IEEE

***Abstract*—In this paper, we show that the Laplace transform of the expansion $h(t) = \sum_{n=0}^{\infty} c_n g_n(t)$ for some important sets $g_n(t)$ is equivalent to a power-series expansion. Techniques based on this result are presented for obtaining the coefficients c_n as those of a power series; also, methods are presented for obtaining the functional series inverse. The set of Laguerre functions is discussed in detail and, using the power-series equivalence, the truncation error is obtained. The application of the power-series equivalence to the summing of series is shown and illustrated with the Neumann series. Finally, the extension of the power-series equivalence to the expansion of functions of several variables is given. The areas for which the techniques developed are relevant include the analysis and design of signals and the identification and synthesis of processes and systems.**

Manuscript received May 1, 1969; revised August 21, 1969.
The author is with the Department of Electrical Engineering, Northeastern University, Boston, Mass.

I. Introduction

In ANALYSIS, it often is desirable to expand a given function $h(t)$ in terms of a set of functions $g_n(t)$ as

$$h(t) = \sum_{n=0}^{\infty} c_n g_n(t). \tag{1}$$

If the set is orthogonal, then the coefficients c_n can be determined individually by integration. However, the integrals may not be simple to evaluate and properties among the coefficients c_n not evident. For some important sets of functions, however, the Laplace transform of (1) is equivalent to a power series. The equivalence is presented in this paper. Its significance is that the well-